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AN

ELEMENTARY TREATISE

ON

PLANE AND SPHERICAL TRIGONOMETRY,

WITH THEIR APPLICATIONS TO

NAVIGATION, SURVEYING, HEIGHTS AND DISTANCES, AND SPHERICAL ASTRONOMY,

AND PARTICULARLY ADAPTED TO EXPLAINING

THE CONSTRUCTION OF BOWDITCH'S NAVIGATOR, AND THE NAUTICAL ALMANAC.

BY

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ERRATA.

Page 320, line 13, for sun's read earth's.

" 321, line 12, for (fig. 17) read (fig. 57).

" 324, line 19, for cot. *ZSN* read tan. *ZSN*.

" 324, line 20, for cot. read tan.

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PLANE TRIGONOMETRY.

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PLANE TRIGONOMETRY.

CHAPTER I.

GENERAL PRINCIPLES OF PLANE TRIGONOMETRY.

1. *Trigonometry* is the science which treats of angles and triangles.

2. *Plane Trigonometry* treats of plane triangles. [B. p. 36.*]

3. *To solve a Triangle* is to calculate certain of its sides and angles when others are known.

It has been proved in Geometry that, when three of the six parts of a triangle are given, the triangle can be constructed, provided one at least of the given parts is a side. In these cases, then, the unknown parts of the triangle can be determined geometrically, and it may readily be inferred that they can also be determined algebraically.

But a great difficulty is met with on the very threshold of the attempt to apply the calculus to triangles. It arises from the circumstance, that two kinds of quantities are to be introduced into the same formulas, sides, and angles. These quantities are not only of an entirely different species, but the law of their relative increase and decrease is so complicated, that they cannot be determined from each other by any of the common operations of Algebra.

---

* References between brackets, preceded by the letter B., refer to the pages in the stereotype edition of Bowditch's Navigator.

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4. To diminish the difficulty of solving triangles as much as possible, every method has been taken to compare triangles with each other, and the solution of all triangles has been reduced to that of a *Limited Series of Right Triangles.*

a. It is a well known proposition of Geometry, that, in all triangles, which are equiangular with respect to each other, the ratios of the homologous sides are also equal. [B. p. 12.] If, then, a series of dissimilar triangles were constructed containing every possible variety of angles; and, if the angles and the ratios of the sides were all known, we should find it easy to calculate every case of triangles. Suppose, for instance, that in the triangle ABC (fig. 1), the sides of which we shall denote by the small letters a, b, c, respectively opposite to the angles A, B, C, there are given the two sides b and c and the included angle A, to find the side a and the angles B and C. We are to look through the series of calculated triangles, till we find one which has an angle equal to A, and the ratio of the including sides equal to that of b and c. As this triangle is similar to ABC, its angles and the ratio of its sides must also be those of the triangle ABC, which is therefore completely determined. For, to find the side a, we have only to multiply the ratio which we have found of b to a, that is, the fraction $\frac{a}{b}$ by the side b, or the ratio $\frac{a}{c}$ by the side c.

b. A series of calculated triangles is not, however, needed for any other than right triangles. For every oblique triangle is either the sum or the difference of two right triangles; and the sides and angles of the oblique triangle are the same with those of the right triangles, or may be obtained from them by addition or by subtraction. Thus the triangle ABC is the sum (fig. 2), or the difference (fig. 3), of the two right triangles ABP and BPC. In both figures the sides AB, BC, and the angle A belong at once to the oblique and the right triangles, and so does the angle BCA (fig. 2), or its supplement (fig. 3); while the angle ABC is the sum (fig. 2), or the difference (fig. 3), of ABP and PBC; and the side AC is the sum (fig. 2), or the difference (fig. 3), of AP and PC.

c. But, as even a series of right triangles, which should contain every variety of angle, would be unlimited, it could never be con-

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structed or calculated. Fortunately, such a series is not required; and it is sufficient for all practical purposes to calculate a series in which the successive angles differ only by a minute, or, at least, by a second. The other triangles can be obtained, when needed, by that simple principle of interpolation made use of to obtain the intermediate logarithms from those given in the tables.

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CHAPTER II.

SINES, TANGENTS, AND SECANTS.

5. CONFINING ourselves, for the present, to right triangles, we proceed to introduce some terms, for the purpose of giving simplicity and brevity to our language.

The *Sine* of an angle is the quotient obtained by dividing the leg opposite it in a right triangle by the hypothenuse.

Thus, if we denote (fig. 4) the legs BC and AC by the letters a and b, and the hypothenuse AB by the letter h, we have

$$\text{sin. } A = \frac{a}{h}, \text{ sin. } B = \frac{b}{a}. \tag{1}$$

6. The *Tangent* of an angle is the quotient obtained by dividing the leg opposite it in a right triangle, by the adjacent leg.

Thus, (fig. 4),

$$\text{tang. } A = \frac{a}{b}, \text{ tang. } B = \frac{b}{a}. \tag{2}$$

7. The *Secant* of an angle is the quotient obtained by dividing the hypothenuse by the leg adjacent to the angle.

Thus, (fig. 4),

$$\text{sec. } A = \frac{h}{b}, \text{ sec. } B = \frac{h}{a}. \tag{3}$$

8. The *Cosine*, *Cotangent*, and *Cosecant* of an angle are respectively the sine, tangent, and secant of its complement.

9. *Corollary.* Since the two acute angles of a right triangle are complements of each other, the sine, tangent, and

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secant of the one must be the cosine, cotangent, and cosecant of the other.

Thus, (fig. 4),

$$\left.\begin{aligned}
\text{sin. } A &= \text{cos. } B = \frac{a}{h}\\
\text{cos. } A &= \text{sin. } B = \frac{b}{h}\\
\text{tang. } A &= \text{cotan. } B = \frac{a}{b}\\
\text{cotan. } A &= \text{tang. } B = \frac{b}{a}\\
\text{sec. } A &= \text{cosec. } B = \frac{h}{b}\\
\text{cosec. } A &= \text{sec. } B = \frac{h}{a}
\end{aligned}\right\} \tag{4}$$

10. *Corollary.* By inspecting the preceding equations (4), we perceive that the sine and cosecant of an angle are reciprocals of each other; as are also the cosine and secant, and also the tangent and cotangent.

So that

$$\left.\begin{aligned}
\text{cosec. } A \times \text{sin. } A &= \frac{h}{a} \times \frac{a}{h} = \frac{ah}{ah} = 1\\
\text{sec. } A \times \text{cos. } A &= \frac{h}{b} \times \frac{b}{h} = \frac{bh}{bh} = 1\\
\text{tang. } A \times \text{cotan. } A &= \frac{a}{b} \times \frac{b}{a} = \frac{ab}{ab} = 1
\end{aligned}\right\} \tag{5}$$

whence

$$\left.\begin{aligned}
\text{cosec. } A &= \frac{1}{\text{sin. } A}, \text{ or sin. } A = \frac{1}{\text{cosec. } A}\\
\text{sec. } A &= \frac{1}{\text{cos. } A}, \text{ or cos. } A = \frac{1}{\text{sec. } A}\\
\text{cotan. } A &= \frac{1}{\text{tan. } A}, \text{ or tang. } A = \frac{1}{\text{cotan. } A}
\end{aligned}\right\} \tag{6}$$

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As soon, then, as the sine, cosine, and tangent of an angle are known, their reciprocals, the cosecant, secant, and cotangent, may easily be obtained.

11. *Problem. To find the tangent when the sine and cosine of an angle are known.*

Solution. The quotient of sin. A divided by cos. A is, by equations (4),

$$\frac{\sin. A}{\cos. A} = \frac{a}{h} \div \frac{b}{h} = \frac{ah}{bh} = \frac{a}{b}.$$

But by (4),

$$\text{tang.}\, A = \frac{a}{b};$$

hence

$$\text{tang.}\, A = \frac{\sin. A}{\cos. A}. \tag{7}$$

12. *Corollary.* Since the cotangent is the reciprocal of the tangent, we have

$$\text{cotan.}\, A = \frac{\cos. A}{\sin. A}. \tag{8}$$

13. *Problem. To find the cosine of an angle when its sine is known.*

Solution. We have, by the Pythagorean proposition, in the right triangle ABC (fig. 4),

$$a^2 + b^2 = h^2.$$

But by (4),

$$(\sin. A)^2 + (\cos. A)^2 = \frac{a^2}{h^2} + \frac{b^2}{h^2} = \frac{a^2 + b^2}{h^2} = \frac{h^2}{h^2} = 1,$$

or
$$(\sin. A)^2 + (\cos. A)^2 = 1; \tag{9}$$

that is, *the sum of the squares of the sine and cosine is equal to unity.*

Hence
$$(\cos. A)^2 = 1 - (\sin. A)^2,$$

$$\cos. A = \sqrt{1 - (\sin. A)^2}. \tag{10}$$

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14. *Corollary.* Since

$$h^2 - a^2 = b^2,$$

we have by (4),

$$(\text{sec. } A)^2 - (\text{tang. } A)^2 = \frac{h^2}{b^2} - \frac{a^2}{b^2} = \frac{h^2 - a^2}{b^2} = \frac{b^2}{b^2} = 1,$$

or $$(\text{sec. } A)^2 - \text{tang. } A)^2 = 1; \qquad (11)$$

whence $$(\text{sec. } A)^2 = 1 + (\text{tang. } A)^2.$$

15. *Corollary.* Since

$$h^2 - b^2 = a^2$$

we have by (4),

$$(\text{cosec. } A)^2 - (\text{cotan. } A)^2 = \frac{h^2}{a^2} - \frac{b^2}{a^2} = \frac{h^2 - b^2}{a^2} = 1,$$

or $$(\text{cosec. } A)^2 - (\text{cotan. } A)^2 = 1; \qquad (12)$$

whence $$(\text{cosec. } A)^2 = 1 + (\text{cotan. } A)^2.$$

16. *Scholium.* The whole difficulty of calculating the trigonometric tables of sines and cosines, tangents and cotangents, secants and cosecants is, by the preceding propositions, reduced to that of calculating the sines alone.

17. Examples.

1. Given the sine of the angle A, equal to 0.4568, calculate its cosine, tangent, cotangent, secant, and cosecant.

Solution. By equation (10)

$$\text{cos. } A = \sqrt{1 - (\text{sin. } A)^2} = \sqrt{(1 + \text{sin. } A)\,(1 - \text{sin. } A)}.$$

$1 + \text{sin. } A = 1.4568$	0.16340
$1 - \text{sin. } A = 0.5432$	9.73496
$(\text{cos. } A)^2$	2|9.89836
$\text{cos. } A = 0.8896$	9.94918

By (7) and (8),

$$\text{tang. } A = \frac{\text{sin. } A}{\text{cos. } A}, \quad \text{cotan. } A = \frac{\text{cos. } A}{\text{sin. } A}.$$

sin. $A = 0.4568$	9.65973 (ar. co.)	10.34027
cos. $A = 0.8896$ (ar. co.)	10.05082	9.94918
tang. $A = 0.5135$	9.71055 (ar. co.)	10.28945

cotan. $A = 1.9474$.

By (6),

$$\text{sec. } A = \frac{1}{\text{cos. } A}, \qquad \text{cosec. } A = \frac{1}{\text{sin. } A}$$

$$\text{log. sec. } A = - \text{ log. cos. } A = 0.05082,$$

$$\text{sec. } A = 1.1241,$$

$$\text{log. cosec. } A = - \text{ log. sin. } A = 0.34027,$$

$$\text{cosec. } A = 2.1891.$$

2. Given sin. $A = 0.1111$; find the cosine, tangent, cotangent, secant, and cosecant of A.

Ans. cos. $A = 0.9938$
tang. $A = 0.1118$
cotan. $A = 8.9452$
sec. $A = 1.0062$
cosec. $A = 9.0010$

3. Given sin. $A = 0.9891$; find the cosine, tangent, cotangent, secant, and cosecant of A.

Ans. cos. $A = 8.1472$
tang. $A = 6.6173$
cotan. $A = 0.1489$
sec. $A = 6.7935$
cosec. $A = 1.0110$

18. *Theorem. The sine of an angle is equal to the perpendicular let fall from one extremity of the arc, which measures it in the circle, whose radius is unity, upon the radius passing through the other extremity.*

Proof. Let BCA (fig. 5) be the angle, and let the radius of the circle $ABA'A$ be

$$AC = \text{unity} = 1.$$

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Let fall, on the radius AC, the perpendicular BP, and we have by § 5, in the right triangle BCP,

$$\sin. BCP = \frac{BP}{BC} = \frac{BP}{1} = BP.$$

19. *Theorem. In the circle of which the radius is unity, the cosine of an angle is equal to the portion of the radius, which is drawn perpendicular to the sine, included between the sine and the centre.*

Proof. For if BCA (fig. 5) is the angle, we have, by § 9,

$$\cos. BCA = \frac{CP}{BC} = \frac{CP}{1} = CP.$$

20. *Theorem. In the circle of which the radius is unity, the secant is equal to the length of the radius drawn through one extremity of the arc which measures the angle, and produced till it meets the tangent drawn through the other extremity.*

The trigonometric tangent is equal to that portion of the tangent, drawn through one extremity of the arc, which is intercepted between the two radii which terminate the arc.

Proof. If CB (fig. 5) is produced to meet the tangent A at T, we have, by (2) and (3), in the right triangle ACT,

$$\sec. BCA = \frac{CT}{AC} = \frac{CT}{1} = CT$$

$$\text{tang. } BCA = \frac{AT}{AC} = \frac{AT}{1} = AT.$$

21. *Scholium.* The preceding theorems (18–20), have been adopted by most writers upon trigonometry, as the definitions of sine, cosine, tangent, and secant, except that the radius of the circle has not been limited to unity. [B. p. 6.]

By not limiting the radius to unity, the sines, &c. have not been fixed values, but have varied with the length of the radius; whereas their values, in the system here adopted, are the fixed ratios of their

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values as ordinarily given to the radius of the circle in which they are measured. Thus, if R is the radius, we have

$$\text{sin., cos., \&c. in the common system} = R \times \text{sin., cos., \&c. in this system.}$$

22. *Corollary.* If the angle is very small, as C (fig. 6), the arc AB will be sensibly a straight line, perpendicular to the two radii CA and CB, drawn to its extremities, and will sensibly coincide with the sine and tangent; while the cosine will sensibly coincide with the radius CA, and the secant with the radius CB.

Hence, the sine and tangent of a very small angle are nearly equal to the arc which measures the angle, in the circle the radius of which is unity; and its cosine and secant are nearly equal to unity.

23. *Problem. To find the sine of a very small angle.*

Solution. Let the angle C (fig. 6) be the given angle, and suppose it to be exactly one minute. The arc AB must in this case be $\frac{1}{10800}$ of the semicircumference, of which unity or CA is radius. But the value of the semicircumference, of which unity is radius, has been found in Geometry to be 3.1415926. Therefore, by § 22,

$$\sin.\ 1' = AB = \frac{3.1415926}{10800} = 0.000290888. \qquad (13)$$

In the same way we might find the sine of any other small angle, or we might, in preference, find it by the following proposition.

24. *Theorem. The sines of very small angles are proportional to the angles themselves.*

Proof. Let there be the two small angles, BCA and $B'CA$ (fig. 7). Draw the arc ABB' with the centre C, and the radius unity. Then, as angles are proportional to the arcs which measure them,

$$BCA : B'CA = BA : B'A.$$

But, by § 22,

$$\sin.\ BCA = BA,\ \sin.\ B'CA = B'A\ ;$$

whence

$$BCA : B'CA = \sin.\ BCA : \sin.\ B'CA.$$

a. This proposition is limited to angles so small, that their arcs may be considered as straight lines. It is found in practice, that the angles may be as large as two degrees, provided the approximations are not carried beyond five places of decimals. The investigation of the sines of larger angles requires the introduction of some new formulas.

25. *Corollary.* It follows from the preceding theorem, that

$$\sin. x = x \sin. 1', \tag{14}$$

provided that x in the second member is expressed in minutes. But if x is expressed in seconds, we have

$$\sin. x = x \sin. 1''; \tag{15}$$

and if x is expressed in terms of the arc which measures it in the circle, whose radius is unity,

$$\sin. x = x; \tag{16}$$

and each of these different notations may be used at pleasure.

26. Examples.

1. Find the sine of $12' \, 13''$, knowing that

$$\sin. 1' = 0.00029.$$

Solution. By (14)

$$1' : 12' \, 13'' :: \sin. 1' : \sin. 12' \, 13'',$$

or

$$60'' : 733'' :: 0.00029 : \sin. 12' \, 13''.$$

Hence

$$\sin. 12' \, 13'' = \frac{733 \times 0.00029}{60} = 0.00354. \qquad \textit{Ans.}$$

2. Find the sine of $7' \, 15''$, knowing that

$$\sin. 1' = 0.00029.$$

Ans. $\sin. 7' \, 15'' = 0.00210.$

3. Find the sine of $2' \, 31''$, knowing that

$$\sin. 1' = 0.00029.$$

Ans. $\sin. 2' \, 31'' = 0.00073.$

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27. *Problem. Given the sine of any angle, to find the sine of another angle which exceeds it by a very small quantity.*

Solution. Let the given angle be BCA (fig. 8), which we will denote by the letter M; and let the angle whose sine is required be $B'CA$, exceeding the former by the small angle $B'CB$, which we will denote by the letter m; so that

$$M = BCA, \qquad m = B'CB,$$
$$M + m = B'CA.$$

From the vertex C as a centre, with the radius unity, describe the arc ABB'. From the points B and B', let fall BP and $B'P'$ perpendicular to AC.

We have, by § 18 and 19,

$$\sin. M = BP$$
$$\sin. B'CA = \sin. (M + m) = B'P'$$
$$\cos. M = PC;$$

Draw BR perpendicular to $B'P'$, and

$$B'P' = BP + B'R,$$

or

$$\sin. (M. + m) = \sin. M + B'R.$$

The triangles BCP and $BB'R$, having their sides perpendicular each to each, are similar, and give the proportion

$$BC : BB' = CP : B'R.$$

But, by § 22,

$$BB' = \sin. m.$$

Hence

$$1 : \sin. m = \cos. M : B'R;$$
$$\text{and } B'R = \sin. m \,.\, \cos. M,$$

which gives, by substitution,

$$\sin. (M + m) = \sin. M + \sin. m \,.\, \cos. M. \tag{17}$$

28. *Corollary.* If m were $1'$, (17) would become by (13)

$$\sin. (M + 1') = \sin. M + \sin. 1' \,.\, \cos. M,$$
$$= \sin. M + 0.00029 \cos. M. \tag{18}$$

We may, by this formula, find the sine of $2'$ from that of $1'$,

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thence that of 3′, then of 4′, of 5′, &c. to the sine of an angle of any number of degrees and minutes.

29. *Corollary.* We can, in a similar way, deduce the value of cos. $(M+m)$.

For, by § 19,

$$\cos. (M+m) = P'C = PC - PP',$$
$$= \cos. M - BR.$$

But the similar triangles $BB'R$ and BCP give the proportion

$$BC : BB' = BP : BR,$$

or

$$1 : \sin. m = \sin. M : BR.$$

Hence

$$BR = \sin. m \,.\, \sin. M,$$

whence

$$\cos. (M+m) = \cos. M - \sin. m \,.\, \sin. M, \tag{19}$$

and, if we make $m = 1'$, this equation becomes

$$\cos. (M+1') = \cos. M - \sin. 1' \,.\, \sin. M,$$
$$= \cos. M - 0.00029 \sin. M. \tag{20}$$

30. Examples.

1. Given the sine of 23° 28′ equal to 0.39822, to find the sine of 23° 29′.

Solution. We find the cosine of 23° 28′ by (10) to be

$$\cos. 23° 28' = 0.91729.$$

Hence, by (18), making $M = 23° 28'$

$$\sin. 23° 29' = \sin. 23° 28' + 0.00029 \cos. 23° 28',$$
$$= 0.39822 + 0.00026,$$
$$= 0.39848.$$

Ans. $\sin. 23° 29' = 0.39848.$

2. Given the sine and cosine of 46° 58′ as follows,

$$\sin. 46° 58' = 0.73096, \cos. 46° 58' = 0.68242,$$

find the sine and cosine of 46° 59′.

Ans. $\sin. 46° 59' = 0.73116,$
$\cos. 46° 59' = 0.68221.$

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3. Given the sine and cosine of $11^\circ\ 10'$ as follows,

$$\sin.\ 11^\circ\ 10' = 0.19366, \quad \cos.\ 11^\circ\ 10' = 0.98107,$$

find the sine and cosine of $11^\circ\ 11'$.

$$\sin.\ 11^\circ\ 11' = 0.19395.$$

$$\textit{Ans.}\quad \cos.\ 11^\circ\ 11' = 0.98101.$$

31. By the formulas here given, a complete table of sines and cosines might be calculated. Such tables have been actually calculated; and table XXIV. of the Navigator is such a table; their logarithms are given in table XXVII. of the Navigator.

The sines, cosines, &c. of table XXIV. are called *natural*, to distinguish them from their logarithms, which are sometimes called their *artificial* sines, cosines, &c.

The radius of table XXIV. is

$$10^5 = 100000,$$

so that this table is, by § 21, reduced to the present system by dividing each number by 100000, that is, by prefixing the decimal point to each of the numbers of the table.

The radius of table XXVII. is

$$10^{10} = 10{,}000{,}000{,}000,$$

so that this table is reduced to the present system by subtracting from each number the logarithm of this radius, which is 10, that is, by subtracting 10 from each characteristic.

The method of using these two tables is fully explained in pp. 33–35, and p. 390, of the Navigator.

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CHAPTER III.

RIGHT TRIANGLES.

32. *Problem. To solve a right triangle, when the hypothenuse and one of the angles are known.* [B. p. 38.]

Solution. Given (fig. 4) the hypothenuse h and the angle A, to solve the triangle.

First. To find the other acute angle B, subtract the given angle from 90°.

Secondly. To find the opposite side a, we have by (1),

$$\sin. A = \frac{a}{h},$$

which, multiplied by h, gives

$$a = h \sin. A; \tag{21}$$

or, by logarithms,

$$\log. a = \log. h + \log. \sin. A.$$

Thirdly. To find the side b, we have by (4)

$$\cos. A = \frac{b}{h},$$

which, multiplied by h, gives

$$b = h \cos. A; \tag{22}$$

or, by logarithms,

$$\log. b = \log. h + \log. \cos. A.$$

33. *Problem. To solve a right triangle, when a leg and the opposite angle are known.* [B. p. 39.]

Solution. Given (fig. 4) the leg a, and the opposite angle A, to solve the triangle.

First. The angle B is the complement of A.

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Secondly. To find the hypothenuse h, we have by (21)

$$a = h \sin. A,$$

which, divided by sin. A, gives by (6)

$$h = \frac{a}{\sin. A} = a \operatorname{cosec.} A; \tag{23}$$

or, by logarithms,

$$\begin{aligned} \log. h &= \log. a + (\text{ar. co.}) \log. \sin. A \\ &= \log. a + \log. \operatorname{cosec.} A. \end{aligned}$$

Thirdly. To find the other leg b, we have by (4)

$$\operatorname{cotan.} A = \frac{b}{a},$$

which, multiplied by a, gives

$$b = a \operatorname{cotan.} A; \tag{24}$$

or, by logarithms,

$$\log. b = \log. a + \log. \operatorname{cotan.} A.$$

34. *Problem. To solve a right triangle, when a leg and the adjacent angle are known.* [B. p. 39.]

Solution. Given (fig. 4) the leg a and the angle B, to solve the triangle.

First. The angle A is the complement of B.

Secondly. The other parts may be found by (23) and (24), or from the following equations, which are readily deduced from equations (4) and (6),

$$h = \frac{a}{\cos. B.} = a \sec. B, \tag{25}$$

$$b = a \operatorname{tang.} B; \tag{26}$$

or, by logarithms,

$$\begin{aligned} \log. h &= \log. a + \log. \sec. B, \\ \log. b &= \log. a + \log. \operatorname{tang.} B. \end{aligned}$$

35. *Problem. To solve a right triangle, when the hypothenuse and a leg are known.* [B. p. 49.]

Solution. Given (fig. 4) the hypothenuse h and the leg a, to solve the triangle.

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First. The angles A and B are obtained from equation (4),

$$\sin. A = \cos. B = \frac{a}{h}; \tag{27}$$

or, by logarithms,

$$\log. \sin A = \log. \cos. B = \log. a + (\text{ar. co.}) \log. h.$$

Secondly. The leg b is deduced from the Pythagorean property of the right triangle, which gives

$$a^2 + b^2 = h^2, \tag{28}$$

whence

$$b^2 = h^2 - a^2 = (h + a)(h - a),$$

$$b = \sqrt{(h^2 - a^2)} = \sqrt{[(h + a)(h - a)]}; \tag{29}$$

by logarithms,

$$\log. b = \tfrac{1}{2} \log. (h^2 - a^2) = \tfrac{1}{2} [\log. (h + a) + \log. (h - a)].$$

36. *Problem. To solve a right triangle, when the two legs are known.* [B. p. 40.]

Solution. Given (fig. 4) the legs a and b, to solve the triangle.

First. The angles are obtained from (4)

$$\text{tang. } A = \text{cotan. } B = \frac{a}{b}; \tag{30}$$

or, by logarithms,

$$\log. \text{tang. } A = \log. \text{cotan. } B = \log. a + (\text{ar. co.}) \log. b.$$

Secondly. To find the hypothenuse, we have by (28)

$$h = \sqrt{(a^2 + b^2)}. \tag{31}$$

Thirdly. An easier way of finding the hypothenuse is to make use of (23) or (25)

$$h = a \text{ cosec. } A = a \text{ sec. } B; \tag{32}$$

or, by logarithms,

$$\log. h = \log. a + \log. \text{cosec. } A = \log. a + \log. \text{sec. } B.$$

37. Examples.

1. Given the hypothenuse of a right triangle equal to 49.58, and one of the acute angles equal to $54° 44'$; to solve the triangle.

Solution. The other angle $= 90° - 54° 44' = 35° 16'$. Then making $h = 49.58$, and $A = 54° 44'$; we have, by (21) and (22),

$h = 49.58$	1.69531		1.69531
$A = 54°44'$	*sin. 9.91194		cos. 9.76146
$a = 40.481$	1.60725 ;	$b = 28.627$	1.45677

Ans. The other angle $= 35°\ 16'$;

$$\text{The legs} = \begin{cases} 40.481 \\ 28.627 \end{cases}$$

2. Given the hypothenuse of a right triangle equal to 54.571, and one of the legs equal to 23.479; to solve the triangle.

Solution. Making $h = 54.571$, $a = 23.479$; we have, by (27),

$a = 23.479$		1.37068
$h = 54.571$	(ar. co.)	8.26304
$A = 25°\ 29'$ sin. $B = 64°\ 31'$ cos.		9.63372

By (29),

$h + a = 78.050$		1.89237
$h - a = 31.092$		1.49265
b^2	2	3.38502
$b = 49.262$		1.69251

Ans. The other leg $= 49.262$

$$\text{The angles} = \begin{cases} 25°\ 29' \\ 64°\ 31' \end{cases}$$

3. Given the two legs of a right triangle equal to 44.375, and 22.165; to solve the triangle.

Solution. Making $a = 44.375$, $b = 22.165$; we have

$a = 44.375$		1.64714		1.64714
$b = 22.165$	(ar. co.)	8.65433		
$A = 63°\ 27'\ 28''$ tang. $B = 26°\ 32'\ 32''$ cotan.		10.30147 ;	cosec. sec.	10.04837
			$h = 49.603$	1.69551

Ans. The hypothenuse $= 49.603$

$$\text{The angles} = \begin{cases} 63°\ 27'\ 28'' \\ 26°\ 32'\ 32'' \end{cases}$$

* To avoid negative characteristics, the logarithms are retained as in the tables, according to the usual practice with the logarithms of decimals, as in B., p. 29.

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4. Given the hypothenuse of a right triangle equal to 37.364, and one of the acute angles equal to 12° 30′; to solve the triangle.

Ans. The other angle = 77° 30′

$$\text{The legs} = \begin{cases} 8.087 \\ 36.478 \end{cases}$$

5. Given one of the legs of a right triangle equal to 14.548, and the opposite angle equal to 54° 24′; to solve the triangle.

Ans. The hypothenuse = 17.892
The other leg = 10.415
The other angle = 35° 36′

6. Given one of the legs of a right triangle equal to 11.111, and the adjacent angle equal to 11° 11′; to solve the triangle.

Ans. The hypothenuse = 11.326
The other leg = 2.197
The other angle = 78° 49′

7. Given the hypothenuse of a right triangle equal to 100, and one of the legs equal to 1; to solve the triangle.

Ans. The other leg = 99.995

$$\text{The angles} = \begin{cases} 0^\circ\ 34'\ 23'' \\ 89^\circ\ 25'\ 37'' \end{cases}$$

8. Given the two legs of a right triangle equal to 8.148, and 10.864; to solve the triangle.

Ans. The hypothenuse = 13.58

$$\text{The angles} = \begin{cases} 36^\circ\ 52'\ 11'' \\ 53^\circ\ \ 7'\ 49'' \end{cases}$$

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CHAPTER IV.

GENERAL FORMULAS.

38. The solution of oblique triangles requires the introduction of several trigonometrical formulas, which it is convenient to bring together and investigate all at once.

39. *Problem. To find the sine of the sum of two angles.*

Solution. Let the two angles be BAC and $B'AC$ (fig. 9), represented by the letters M and N. At any point C, in the line AC, erect the perpendicular BB'. From B let fall on AB' the perpendicular BP. Then represent the several lines as follows,

$$a = BC,\ a' = B'C,\ b = AC$$
$$h = AB,\ h' = AB',\ x = BP$$
$$M = BAC,\ N = B'AC.$$

Then, by (4),

$$\sin.\ BAC = \sin.\ M = \frac{a}{h}, \qquad \sin.\ N = \frac{a'}{h'}$$

$$\cos.\ M = \frac{b}{h}, \qquad \cos.\ N = \frac{b}{h'}$$

$$\sin.\ BAP = \sin.\ (M + N) = \frac{BP}{AB} = \frac{x}{h}.$$

Now the triangles BPB' and $B'AC$, being right-angled, and having the angle B' common, are equiangular and similar.

Whence we derive the proportion

$$AB' : BB' = AC : BP,$$

or

$$h' : a + a' = b : x;$$

whence

$$x = \frac{a\,b + a'\,b}{h'},$$

and

$$\sin.\ (M + N) = \frac{x}{h} = \frac{a\,b + a'\,b}{h\,h'}.$$

The second member of this equation may be separated into factors, as follows,

$$\sin. (M + N) = \frac{a\,b}{h\,h'} + \frac{b\,a'}{h\,h'}$$

$$= \frac{a}{h} \cdot \frac{b}{h'} + \frac{b}{h} \cdot \frac{a'}{h'};$$

whence, by substitution, we obtain

$$\sin. (M + N) = \sin. M \cos. N + \cos. M \sin. N. \qquad (33)$$

40. *Problem. To find the sine of the difference of two angles.*

Solution. Let the two angles be BAC and $B'AC$ (fig. 10), represented by M and N. At any point C in the line AC erect the perpendicular $BB'C$. From B let fall on AB' the perpendicular BP. Then, using the notation of § 39, we have

$$\sin. BAP = \sin. (M - N) = \frac{BP}{AB} = \frac{x}{h}.$$

The triangles $B'AC$ and $BB'P$ are similar, because they are right-angled, and the angles at B' are vertical and equal.

Whence

$$AB' : BB = AC : BP,$$

or

$$h' : a - a' = b : x;$$

whence

$$x = \frac{a\,b - a'\,b}{h'},$$

and

$$\sin. (M - N) = \frac{x}{h} = \frac{a\,b - b\,a'}{h\,h'}$$

$$= \frac{a\,b}{h\,h'} - \frac{b\,a'}{h\,h'}$$

$$= \frac{a}{h} \cdot \frac{b}{h'} - \frac{b}{h} \cdot \frac{a'}{h};$$

and by substitution,

$$\sin. (M - N) = \sin. M \cos. N - \cos. M \sin. N. \qquad (34)$$

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41. *Problem. To find the cosine of the sum of two angles.*

Solution. Making use of (fig. 9), with the notation of § 39, and also the following

$$y = AP,\ z = PB';$$

we have

$$\cos.\ (M + N) = \frac{AP}{AB} = \frac{y}{h}.$$

But

$$y = AB' - PB' = h' - z.$$

The similar triangles BPB' and $B'AC$, give the proportion

$$AB' : BB' = B'C : B'P,$$

or

$$h' : a + a' = a' : z;$$

whence

$$z = \frac{a\,a' + a'^2}{h'},$$

and

$$y = h' - z = h' - \frac{a\,a' + a'^2}{h'}$$

$$= \frac{h'^2 - a'^2 - a\,a'}{h'}.$$

But, from the right triangle $AB'C$,

$$h'^2 - a'^2 = (AB')^2 - (B'C)^2 = (AC)^2 = b^2;$$

whence

$$y = \frac{b^2 - a\,a'}{h'}$$

and

$$\cos.\ (M + N) = \frac{y}{h} = \frac{b^2 - a\,a'}{h\,h'}$$

$$= \frac{b^2}{h\,h'} - \frac{a\,a'}{h\,h'},$$

$$= \frac{b}{h} \cdot \frac{b}{h'} - \frac{a}{h} \cdot \frac{a'}{h'};$$

whence by substitution,

$$\cos.\ (M + N) = \cos.\ M \cos.\ N - \sin.\ M \sin.\ N. \tag{35}$$

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42. *Problem. To find the cosine of the difference of two angles.*

Solution. Making use of (fig. 10) with the notation of the preceding section, we have

$$\cos. BAB' = \cos. (M - N) = \frac{AP}{AB} = \frac{y}{h}.$$

But $$y = AB' + PB' = h' + z.$$

The similar triangles $BB'P$ and $B'AC$ give the proportion

$$AB' : BB' = B'C : B'P,$$

or $$h' : a - a' = a' : z;$$

whence $$z = \frac{a\,a' - a'^2}{h'},$$

and $$y = h' + z = h' + \frac{a\,a' - a'^2}{h'}$$

$$= \frac{h'^2 - a'^2 + a\,a'}{h'}.$$

But $$h'^2 - a'^2 = b^2.$$

Hence $$y = \frac{b^2 + a\,a'}{h'},$$

and $$\cos. (M - N) = \frac{y}{h} = \frac{b^2 + a\,a'}{h\,h'}$$

$$= \frac{b^2}{h\,h'} + \frac{a\,a'}{h\,h'}$$

$$= \frac{b}{h} \cdot \frac{b}{h'} + \frac{a}{h} \cdot \frac{a'}{h'};$$

or, by substitution,

$$\cos. (M - N) = \cos. M \cos. N + \sin. M \sin. N. \tag{36}$$

43. *Corollary.* The similarity, in all but the signs, of the formulas (33) and (34) is such, that they may both be written in the same form, as follows,

$$\sin. (M \pm N) = \sin. M \cos. N \pm \cos. M \sin. N, \tag{37}$$

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in which the upper signs correspond with each other, and also the lower ones.

In the same way, by the comparison of (35) and (36), we are led to the form

$$\cos. (M \pm N) = \cos. M \cos. N \mp \sin. M \sin. N, \qquad (38)$$

in which the upper signs correspond with each other, and also the lower ones.

44. *Corollary.* The sum of the equations (33) and (34) is

$$\sin. (M + N) + \sin. (M - N) = 2 \sin. M \cos. N. \qquad (39)$$

Their difference is

$$\sin. (M + N) - \sin. (M - N) = 2 \cos. M \sin. N. \qquad (40)$$

45. *Corollary.* The sum of (35) and (36) is

$$\cos. (M + N) + \cos. (M - N) = 2 \cos. M \cos. N. \qquad (41)$$

Their difference is

$$\cos. (M - N) - \cos. (M + N) = 2 \sin. M \sin. N. \qquad (42)$$

46. *Corollary.* If, in (39–42), we make

$$M + N = A, \text{ and } M - N = B;$$

that is,

$$M = \tfrac{1}{2} (A + B), \qquad N = \tfrac{1}{2} (A - B);$$

they become, as follows,

$$\sin. A + \sin. B = 2 \sin. \tfrac{1}{2} (A + B) \cos. \tfrac{1}{2} (A - B) \qquad (43)$$

$$\sin. A - \sin. B = 2 \cos. \tfrac{1}{2} (A + B) \sin. \tfrac{1}{2} (A - B) \qquad (44)$$

$$\cos. A + \cos. B = 2 \cos. \tfrac{1}{2} (A + B) \cos. \tfrac{1}{2} (A - B) \qquad (45)$$

$$\cos. B - \cos. A = 2 \sin. \tfrac{1}{2} (A + B) \sin. \tfrac{1}{2} (A - B). \qquad (46)$$

47. *Corollary.* The quotient, obtained by dividing (43) by (44), is

$$\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{\sin. \tfrac{1}{2} (A + B) \cos. \tfrac{1}{2} (A - B)}{\cos. \tfrac{1}{2} (A + B) \sin. \tfrac{1}{2} (A - B)}.$$

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Reducing the second member by means of equations (6), (7), (8), we have

$$\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \text{tang. } \tfrac{1}{2} (A + B) \text{ cotan. } \tfrac{1}{2} (A - B)$$

$$= \frac{\text{tang. } \frac{1}{2} (A + B)}{\text{tang. } \frac{1}{2} (A - B)} = \frac{\text{cotan. } \frac{1}{2} (A - B)}{\text{cotan. } \frac{1}{2} (A + B)}. \tag{47}$$

48. *Corollary.* The quotient of (46) divided by (45) is, by reduction,

$$\frac{\cos. B - \cos. A}{\cos. B + \cos. A} = \text{tang. } \tfrac{1}{2} (A + B) \text{ tang. } \tfrac{1}{2} (A - B)$$

$$= \frac{\text{tang. } \frac{1}{2} (A + B)}{\text{cotan. } \frac{1}{2} (A - B)} = \frac{\text{tang. } \frac{1}{2} (A - B)}{\text{cotan. } \frac{1}{2} (A + B)}. \tag{48}$$

49. *Corollary.* Putting in (33) and (35), M and N each equal to A, we obtain

$$\sin. 2 A = \sin. A \cos. + \sin. A \cos. A = 2 \sin. A \cos. A \tag{49}$$

$$\cos. 2 A = \cos. A \cos. A - \sin. A \sin. A$$

$$= (\cos. A)^2 - (\sin. A)^2. \tag{50}$$

50. *Corollary.* The sum of (50), and of the following equation, which is the same as (9),

$$1 = (\cos. A)^2 + (\sin. A)^2,$$

is

$$1 + \cos. 2 A = 2 (\cos. A)^2. \tag{51}$$

Their difference is

$$1 - \cos. 2 A = 2 (\sin. A)^2. \tag{52}$$

51. *Corollary.* Making $2 A = C$, or $C = \frac{1}{2} A$, in (49-52), we obtain

$$\sin. C = 2 \sin. \tfrac{1}{2} C \cos. \tfrac{1}{2} C \tag{53}$$

$$\cos. C = (\cos. \tfrac{1}{2} C)^2 - (\sin. \tfrac{1}{2} C)^2 \tag{54}$$

$$1 + \cos. C = 2 (\cos. \tfrac{1}{2} C)^2 \tag{55}$$

$$1 - \cos. C = 2 (\sin. \tfrac{1}{2} C)^2. \tag{56}$$

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The equations (55) and (56) give

$$\cos.\ \tfrac{1}{2}\,C = \sqrt{\left[\tfrac{1}{2}\,(1 + \cos.\ C)\right]} \tag{57}$$

$$\sin.\ \tfrac{1}{2}\,C = \sqrt{\left[\tfrac{1}{2}\,(1 - \cos.\ C)\right]} \tag{58}$$

$$\text{tang.}\ \tfrac{1}{2}\,C = \sqrt{\left(\frac{1 - \cos.\ C}{1 + \cos.\ C}\right)}. \tag{59}$$

52. *Problem. To find the tangent of the sum and of the difference of two angles.*

Solution. First. To find the tangent of the sum of two angles, which we will suppose to be M and N, we have from (7),

$$\text{tang.}\ (M + N) = \frac{\sin.\ M\,(+\ N)}{\cos.\ (M + N)}.$$

Substituting (33) and (35),

$$\text{tang.}\ (M + N) = \frac{\sin.\ M \cos.\ N + \cos.\ M . \sin.\ N}{\cos.\ M \cos.\ N - \sin.\ M . \sin.\ N}.$$

Divide every term of both numerator and denominator of the second member by cos. M cos. N;

$$\text{tang.}\ (M + N) = \frac{\dfrac{\sin.\ M \cos.\ N}{\cos.\ M \cos.\ N} + \dfrac{\cos.\ M \sin.\ N}{\cos.\ M \cos.\ N}}{\dfrac{\cos.\ M \cos.\ N}{\cos.\ M \cos.\ N} - \dfrac{\sin.\ M \sin.\ N}{\cos.\ M \cos.\ N}},$$

$$= \frac{\dfrac{\sin.\ M}{\cos.\ M} + \dfrac{\sin.\ N}{\cos.\ N}}{1 - \dfrac{\sin.\ M}{\cos.\ M} \cdot \dfrac{\sin.\ N}{\cos.\ N}},$$

which, reduced by means of (7), becomes

$$\text{tang.}\ (M + N) = \frac{\text{tang.}\ M + \text{tang.}\ N}{1 - \text{tang.}\ M\ \text{tang.}\ N}. \tag{60}$$

Secondly. To find the tangent of the difference of M and N. Since by (7)

$$\text{tang.}\ (M - N) = \frac{\sin.\ (M - N)}{\cos.\ (M - N)},$$

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a bare inspection of (37) and (38) shows that we have only to change the signs, which connect the terms in the value of tang. $(M + N)$ to obtain that of tang. $(M - N)$. This change, being made in (60), produces

$$\text{tang.}\,(M - N) = \frac{\text{tang.}\,M - \text{tang.}\,N}{1 + \text{tang.}\,M\,\text{tang.}\,N}. \tag{61}$$

53. *Corollary.* As the cotangent is merely the reciprocal of the tangent, we have, by inverting the fractions, from (60) and (61),

$$\text{cotan.}\,(M + N) = \frac{1 - \text{tang.}\,M\,\text{tang.}\,N}{\text{tang.}\,M + \text{tang.}\,N}, \tag{62}$$

$$\text{cotan.}\,(M - N) = \frac{1 + \text{tang.}\,M\,\text{tang.}\,N}{\text{tang.}\,M - \text{tang.}\,N}. \tag{63}$$

54. *Corollary.* Make $M = N = A$, in (60) and (62). They become

$$\text{tang.}\;2\,A = \frac{2\,\text{tang.}\,A}{1 - (\text{tang.}\,A)^2}, \tag{64}$$

$$\text{cotan.}\,2\,A = \frac{1 - (\text{tang.}\,A)^2}{2\,\text{tang.}\,A}. \tag{65}$$

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CHAPTER V.

VALUES OF THE SINES, COSINES, TANGENTS, COTANGENTS, SECANTS, AND COSECANTS OF CERTAIN ANGLES.

55. *Problem. To find the sine, &c. of* 0° *and* 90°.

Solution. It is evident, from § 22, that the sine and tangent of zero are, each of them, zero, while its cosine is unity. Hence, and from the consideration that 0° and 90° are complements of each other, we have

$$\sin. 0^\circ = \cos. 90^\circ = 0. \tag{66}$$

$$\cos. 0^\circ = \sin. 90^\circ = 1. \tag{67}$$

From (6) and (7), we have

$$\text{tang. } 0^\circ = \text{cotan. } 90^\circ = \frac{\sin. 0^\circ}{\cos. 0^\circ} = \tfrac{0}{1} = 0 \tag{68}$$

$$\text{cotan. } 0^\circ = \text{tang. } 90^\circ = \frac{1}{\text{tang. } 0^\circ} = \tfrac{1}{0} = \infty \tag{69}$$

$$\text{sec. } 0^\circ = \text{cosec. } 90^\circ = \frac{1}{\cos. 0^\circ} = \tfrac{1}{1} = 1 \tag{70}$$

$$\text{cosec. } 0^\circ = \text{sec. } 90^\circ = \frac{1}{\sin. 0^\circ} = \tfrac{1}{0} = \infty . \tag{71}$$

56. *Problem. To find the sine, &c. of* 180°.

Solution. Make $A = 90^\circ$, in (49) and (50); they become, by means of (66) and (67),

$$\sin. 180^\circ = 2 \sin. 90^\circ \cos. 90^\circ = 0 \tag{72}$$

$$\cos. 180^\circ = (\cos. 90^\circ)^2 - (\sin. 90^\circ)^2 = -1. \tag{73}$$

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Hence from (6) and (7),

$$\text{tang. } 180^\circ = \frac{\sin. 180^\circ}{\cos. 180^\circ} = \frac{0}{-1} = 0 \tag{74}$$

$$\text{cotan. } 180^\circ = \frac{\cos. 180^\circ}{\sin. 180^\circ} = \frac{-1}{0} = -\infty \tag{75}$$

$$\text{sec. } 180^\circ = \frac{1}{\cos. 180^\circ} = \frac{1}{-1} = -1 \tag{76}$$

$$\text{cosec. } 180^\circ = \frac{1}{\sin. 180^\circ} = \tfrac{1}{0} = \infty. \tag{77}$$

57. *Problem. To find the sine, &c. of* 270°.

Solution. Make $M = 180^\circ$ and $N = 90^\circ$ in (33) and (35). They become, by means of (66, 67, 72, 73),

$$\sin. 270^\circ = \sin. 180^\circ \cos. 90^\circ + \cos. 180^\circ \sin. 90^\circ = -1 \tag{78}$$

$$\cos. 270^\circ = \cos. 180^\circ \cos. 90^\circ - \sin. 180^\circ \sin. 90^\circ = 0. \tag{79}$$

Hence, from (6) and (7),

$$\text{tang. } 270^\circ = \frac{\sin. 270^\circ}{\cos. 270^\circ} = \frac{-1}{0} = -\infty \tag{80}$$

$$\text{cotan. } 270^\circ = \frac{\cos. 270^\circ}{\sin. 270^\circ} = \frac{0}{-1} = 0 \tag{81}$$

$$\text{sec. } 270^\circ = \frac{1}{\cos. 270^\circ} = \tfrac{1}{0} = \infty \tag{82}$$

$$\text{cosec. } 270^\circ = \frac{1}{\sin. 270^\circ} = \frac{1}{-1} = -1. \tag{83}$$

58. *Problem. To find the sine, &c. of* 360°.

Solution. Make $A = 180^\circ$ in (49) and (50); and they become by (72, 73, 66, 67)

$$\sin. 360^\circ = 0 = \sin. 0^\circ \tag{84}$$

$$\cos. 360^\circ = 1 = \cos. 0^\circ. \tag{85}$$

Hence the sine, &c. of 360° *are the same as those of* 0°.

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59. *Problem.* *To find the sine, &c. of* 45°.

Solution. Make $C = 90°$ in (57) and (58). They become, by means of (66),

$$\cos. 45° = \surd[\tfrac{1}{2}(1 + \cos. 90°)] = \surd\tfrac{1}{2} \tag{86}$$

$$\sin. 45° = \surd[\tfrac{1}{2}(1 - \cos. 90°)] = \surd\tfrac{1}{2} = \cos. 45°. \tag{87}$$

Hence, from (6) and (7),

$$\text{tang. } 45° = \frac{\sin. 45°}{\cos. 45°} = 1 \tag{88}$$

$$\text{cotan. } 45° = \frac{1}{\text{tang. } 45°} = 1 = \text{tang. } 45° \tag{89}$$

$$\sec. 45° = \frac{1}{\cos. 45°} = \frac{1}{\surd\frac{1}{2}} = \surd 2 \tag{90}$$

$$\text{cosec. } 45° = \frac{1}{\sin. 45°} = \frac{1}{\surd\frac{1}{2}} = \surd 2 = \sec. 45°. \tag{91}$$

60. *Problem.* *To find the sine, &c. of* 30° *and* 60°.

Solution. Make $A = 30°$ in (49). It becomes, from the consideration that 30° and 60° are complements of each other,

$$\sin. 60° = \cos. 30° = 2 \sin. 30° \cos. 30°.$$

Dividing by cos. 30°, we have

$$1 = 2 \sin. 30°,$$

or

$$\sin. 30° = \tfrac{1}{2} = \cos. 60°; \tag{92}$$

whence, from (6), (7), and (10),

$$\cos. 30° = \sin. 60° = \surd(1 - \tfrac{1}{4}) = \tfrac{1}{2}\surd 3 \tag{93}$$

$$\text{tang. } 30° = \text{cotan. } 60° = \frac{\frac{1}{2}}{\frac{1}{2}\surd 3} = \frac{1}{\surd 3} = \surd\tfrac{1}{3} \tag{94}$$

$$\text{cotan. } 30° = \text{tang. } 60° = \frac{1}{\surd\frac{1}{3}} = \surd 3 \tag{95}$$

$$\sec. 30° = \text{cosec. } 60° = \frac{1}{\frac{1}{2}\surd 3} = \frac{2}{\surd 3} \tag{96}$$

$$\text{cosec. } 30° = \sec. 60° = \frac{1}{\frac{1}{2}} = 2. \tag{97}$$

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61. *Problem. To find the sine, &c. of the supplement of an angle.*

Solution. Make $M = 180°$ in (34) and (36). They become, by means of (72) and (73),

$$(\sin. 180° - N) = \sin. 180° \cos. N - \cos. 180° \sin. N = \sin. N \quad (98)$$

$$\cos. (180° - N) = \cos. 180° \cos. N + \sin. 180° \sin. N = -\cos. N, \quad (99)$$

whence, from (6) and (7),

$$\text{tang. } (180° - N) = -\text{ tang. } N \quad (100)$$

$$\text{cotan. } (180° - N) = -\text{ cotan. } N \quad (101)$$

$$\text{sec. } (180° - N) = -\text{ sec. } N \quad (102)$$

$$\text{cosec. } (180° - N) = \text{ cosec. } N; \quad (103)$$

that is, *the sine and cosecant of the supplement of an angle are the same with those of the angle itself; and the cosine, tangent, cotangent, and secant of the supplement are the negative of those of the angle.*

62. *Corollary.* Since, when an angle is acute its supplement is obtuse, it follows, from the preceding proposition, that *the sine and cosecant of an obtuse angle are positive, while its cosine, tangent, cotangent, and secant are negative.*

This proposition must be carefully borne in mind in using the trigonometric tables, as it affords the means of discriminating between the two angles which are given in B. Table XXVII, and of deciding which of these two angles is the required one.

63. *Corollary.* The preceding corollary might also have been obtained from (33) and (35). For by making $M = 90°$, we have by (66) and (67)

$$\sin. (90° + N) = \cos. N \quad (104)$$

$$\cos. (90° + N) = -\sin. N; \quad (105)$$

whence, by (6) and (7),

$$\text{tang. } (90° + N) = -\text{ cotan. } N \quad (106)$$

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$$\text{cotan.}\,(90^\circ + N) = -\,\text{tang.}\ N \tag{107}$$

$$\text{sec.}\ (90^\circ + N) = -\,\text{cosec.}\ N \tag{108}$$

$$\text{cosec.}\,(90^\circ + N) = \ \text{sec.}\ N; \tag{109}$$

that is, *the sine and cosecant of an angle, which exceeds* 90°, *are equal to the cosine and secant of its excess above* 90°, *while its cosine, tangent, cotangent and secant are equal to the negative of the sine, cotangent, tangent, and cosecant of this excess.*

64. *Problem. To find the sine, &c. of a negative angle.*

Solution. Make $N = 0^\circ$ in (34) and (36). They become, by means of (66) and (67),

$$\text{sin.}\ (-N) = -\,\text{sin.}\ N \tag{110}$$

$$\text{cos.}\ (-N) = \ \text{cos.}\ N; \tag{111}$$

whence, from (6) and (7),

$$\text{tang.}\ (-N) = -\,\text{tang.}\ N \tag{112}$$

$$\text{cotan.}\,(-N) = -\,\text{cotan.}\ N \tag{113}$$

$$\text{sec.}\ (-N) = \ \text{sec.}\ N \tag{114}$$

$$\text{cosec.}\,(-N) = -\,\text{cosec.}\ N; \tag{115}$$

so that *the cosine and secant of the negative of an angle are the same with those of the angle itself; and the sine, tangent, cotangent, and cosecant of the negative of the angle are the negative of those of the angle.*

65. *Problem. To find the sine, &c. of an angle which exceeds* 180°.

Solution. Make $M = 180^\circ$ in (33) and (35). They become, by means of (72) and (73),

$$\text{sin.}\ (180^\circ + N) = -\,\text{sin.}\ N \tag{116}$$

$$\text{cos.}\ (180^\circ + N) = -\,\text{cos.}\ N; \tag{117}$$

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whence, from (6) and (7),

$$\text{tang. } (180^\circ + N) = \text{tang. } N \tag{118}$$

$$\text{cotan. } (180^\circ + N) = \text{cotan. } N \tag{119}$$

$$\text{sec. } (180^\circ + N) = -\text{sec. } N \tag{120}$$

$$\text{cosec. } (180^\circ + N) = -\text{cosec. } N; \tag{121}$$

that is, *the tangent and cotagent of an angle, which exceeds* 180°, *are equal to those of its excess above* 180°; *and the sine, cosine, secant, and cosecant of this angle are the negative of those of its excess.*

66. *Corollary.* If the excess of the angle above 180° is less than 90°, the angle is contained between 180° and 270°; so that *the tangent and cotangent of an angle which exceeds* 180°, *and is less than* 270°, *are positive; while its sine, cosine, secant, and cosecant are negative.*

67. *Corollary.* If the excess of the angle above 180° is greater than 90° and less than 180°, the angle is contained between 270° and 360°; so that, by § 65 and 62, *the cosine and secant of an angle which exceeds* 270° *and is less than* 360°, *is positive; while its sine, tangent, cotangent, and cosecant are negative.*

68. *Corollary.* The results of the two preceding corollaries might have been obtained from (34) and (36). For by making $M = 360^\circ$, we have, by § 58,

$$\text{sin. } (360^\circ - N) = -\text{sin. } N \tag{122}$$

$$\text{cos. } (360^\circ - N) = \text{cos. } N; \tag{123}$$

whence, by (6) and (7),

$$\text{tang. } (360^\circ - N) = -\text{tang. } N \tag{124}$$

$$\text{cotan. } (360^\circ - N) = -\text{cotan. } N \tag{125}$$

$$\text{sec. } (360^\circ - N) = \text{sec. } N \tag{126}$$

$$\text{cosec. } (360^\circ - N) = -\text{cosec. } N; \tag{127}$$

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that is, *the cosine and secant of an angle are the same with those of the remainder after subtracting the angle from* 360°; *while its sine, tangent, cotangent, and cosecant are the negative of those of this remainder.*

69. *Problem. To find the sine, &c. of an angle which exceeds* 360°.

Solution. Make $M = 360^\circ$ in (33) and (35). They become, by means of (84) and (85),

$$\sin. (360^\circ + N) = \sin. N \tag{128}$$

$$\cos. (360^\circ + N) = \cos. N; \tag{129}$$

that is, *the sine, &c. of an angle which exceeds* 360° *are equal to those of its excess above* 360°.

70. *Theorem. The sine, tangent, and secant of an acute angle increase with the increase of the angle; the cosine, cotangent, and cosecant decrease.*

Proof. I. The excess of the sine of $M + m$ over the sine of M is, by (17), equal to $\sin. m \cos. M$, which is a positive quantity when M is acute; and, therefore, the sine of the acute angle increases with the increase of the angle.

II. The excess of $\cos. M$ over $\cos. (M + m)$ is, by (19), equal to $\sin. m \sin. M$, which is a positive quantity; and, therefore, the cosine of the acute angle decreases with the increase of the angle.

III. The tangent of an angle is, by (7), the quotient of its sine divided by its cosine. It is, therefore, a fraction whose numerator increases with the increase of the angle, while its denominator decreases. Either of these changes in the terms of the fraction would increase its value; and, therefore, the tangent of an acute angle increases with the increase of the angle.

IV. The cosecant, secant, and cotangent of an angle are, by (6), the respective reciprocals of the sine, cosine, and tangent. But the reciprocal of a quantity increases with the decrease of the quantity,

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and the reverse. It follows, then, from the preceding demonstrations, that its secant increases with the increase of the acute angle, while its cosecant and cotangent decrease.

71. *Theorem. The absolute values (neglecting their signs) of the sine, tangent and secant of an obtuse angle decrease with the increase of the angle; while those of the cosine, cotangent, and cosecant increase.*

Proof. The supplement of an obtuse angle is an acute angle, of which the absolute values of the sine, &c. are, by § 61, the same as those of the angle itself. But this acute angle decreases with the increase of the obtuse angle, and at the same time its sine, tangent, and secant decrease, while its cosine, cotangent, and cosecant increase.

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CHAPTER VI.

OBLIQUE TRIANGLES.

72. *Theorem. The sides of a triangle are directly proportional to the sines of the opposite angles.* [B. p. 13.]

Proof. In the triangle ABC (figs. 2 and 3), denote the sides opposite the angles A, B, C, respectively, by the letters a, b, c. We are to prove that

$$\sin. A : \sin. B : \sin. C = a : b : c. \tag{130}$$

From the vertex B, let fall on the opposite side the perpendicular BP, which we will denote by the letter p. Then, in the triangle BAP, we have by (1)

$$\sin. A = \frac{BP}{AB} = \frac{p}{c},$$

or

$$p = c \sin. A. \tag{131}$$

Also, in the triangle BPC, we have, by (1) and (98), and from the consideration that BCP is the angle C (fig. 2), and its supplement (fig. 3),

$$\sin. BCP = \sin. C = \frac{BP}{BC} = \frac{p}{a},$$

or

$$p = a \sin. C. \tag{132}$$

Comparing (131) and (132), we have

$$c \sin. A = a \sin. C,$$

which may be converted into the following proportion,

$$\sin. A : \sin. C = a : c.$$

In the same way, it may be proved that

$$\sin. A : \sin. B = a : b;$$

and these two proportions may be written in one as in (130).

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73. *Problem. To solve a triangle, when one of its sides and two of its angles are known.* [B. p. 41.]

Solution. First. The third angle may be found by subtracting the sum of the two given angles from 180°.

Secondly. To find either of the other sides, we have only to make use of a proportion, derived from § 72. As the sine of the angle opposite the given side is to the sine of the angle opposite the required side, so is the given side to the required side. Thus, if a (fig. 1) were the given and b the required side, we should have the proportion

$$\sin. A : \sin. B = a : b;$$

whence by (6)

$$b = \frac{a \sin. B}{\sin. A} = a \sin. B \operatorname{cosec.} A. \tag{133}$$

74. Examples.

1. Given one side of a triangle equal to 22.791, and the adjacent angles equal to 32° 41′ and 47° 54′; to solve the triangle.

Solution. The other angle $= 180° - (32° 41' + 47° 54') = 99° 25'$.

By (133)

99° 25′ cosec.	10.00589		10.00589
32° 41′ sin.	9.73239	47° 54′ sin.	9.87039
22.791	1.35776		1.35776
12.475	*1.09604;	17.141	*1.23404

Ans. The other angle $= 99° 25'$

$$\text{The other sides} = \begin{cases} 12.475 \\ 17.141 \end{cases}$$

* 20 is subtracted from each of these characteristics, because the two sines and cosecant were taken from the tables without any diminution, as required by § 30.

2. Given one side of a triangle equal to 327.06, and the adjacent angles equal to 154° 22′ and 17° 35′; to solve the triangle.

Ans. The other angle = 8° 3′

$$\text{The other sides} = \begin{cases} 1010.4 \\ 705.5 \end{cases}$$

75. *Problem. To solve a triangle, when two of its sides and an angle opposite one of the given sides are known.* [B. p. 42.]

Solution. First. The angle opposite the other given side is found by the proportion of § 72. As the side opposite the given angle is to the other given side, so is the sine of the given angle to the sine of the required angle. Thus, if (fig. 1) a and b are the given sides and A the given angle, the angle B is found, by the proportion

$$a : b = \sin. A : \sin. B;$$

whence

$$\sin. B = \frac{b \sin. A}{a}. \tag{134}$$

Secondly. The third angle is found by subtracting the sum of the two known angles from 180°.

Thirdly. The third side is found by the proportion. As the sine of the given angle is to the sine of the angle opposite the required side, so is the side opposite the given angle to the required side. That is, in the present case,

$$\sin. A : \sin. C = a : c;$$

whence

$$c = \frac{a \sin. C}{\sin. A}, = a \sin. c \operatorname{cosec}. A. \tag{135}$$

76. *Scholium.* Two angles are given in the tables corresponding to the same sine, which are supplements of each other, one being acute and the other obtuse. Two values of B (134) are then given in the tables, and both these values may be possible, when the given value of b is greater than that of a, and the given value of A is less

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than 90° ; for, in this case, there may be two triangles, ABC (fig. 11) and $AB'C$, which satisfy the data.

77. *Scholium.* The problem is impossible, when the given value of b is greater than that of a, and the given value of A is obtuse. For the greater side of an obtuse-angled triangle must always be opposite the obtuse angle.

78. *Scholium.* The problem is impossible, when the given value of b is so much greater than that of a, that we have

$$b \sin. A > a;$$

for, in this case, the given value of a is less than that of the perpendicular CP (fig. 11) from C upon AP.

79. *Scholium.* The obtuse value of B does not satisfy the problem, when b is less than a; for the obtuse angle of a triangle cannot be opposite a smaller side. In this case, therefore, the problem admits of only one solution.

80. Examples.

1. Given two sides of a triangle equal to 77.245 and 92.341, and the angle opposite the first side equal to 55° 28′ 12″; to solve the triangle.

Solution. Making

$$b = 92.341, a = 77.245, A = 55° 28' 12'',$$

we have, by (134),

$a = 77.245$		(ar. co.)	8.11213
$b = 92.341$			1.96540
$A = 55° 28' 12''$		sin.	9.91584
$B = 80° 1'$	or $= 99° 59'$	sin.	9.99337
$A + B = 135° 29' 12''$	or $= 155° 27' 12''$		
$C = 44° 30' 48''$	or $= 24° 32' 48''$		

Then, by (135),

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$a = 77.245$	1.88787		1.88787
$C = 44° 30' 48''$ sin.	9.84576	or $= 24° 32' 48''$ sin.	9.61850
$A = 55° 28' 12''$ cosec.	10.08416		10.08416
$C = 65.734$	1.81779	or $= 38.952$	1.59053

Ans. The third side $=$ 65.734 or $=$ 38.952

The other angles $= \begin{cases} 80° \ 1' \\ 44° \ 30' \ 48'' \end{cases}$ or $= \begin{cases} 99° \ 59' \\ 24° \ 32' \ 48'' \end{cases}$

2. Given two sides of a triangle equal to 77.245 and 92.341, and the angle opposite the second side equal to $55° 28' 12''$; to solve the triangle.

Ans. The third side $=$ 110.7

The other angles $= \begin{cases} 43° \ 33' \ 44'' \\ 80° \ 58' \ 4'' \end{cases}$

3. Given two sides of a triangle equal to 40 and 50, and the angle opposite the first side equal to 45°; to solve the triangle.

Ans. The third side $=$ 54.061 or $=$ 16.65

The other angles $= \begin{cases} 62° \ 7' \\ 72° \ 53' \end{cases}$ or $= \begin{cases} 117° \ 53' \\ 17° \ 7' \end{cases}$

4. Given two sides of a triangle equal to 77.245 and 92.341, and the angle opposite the second side equal to $124° 31' 48''$; to solve the triangle.

Ans. The third side $=$ 23.129.

The other angles $= \begin{cases} 43° \ 33' \ 44'' \\ 11° \ 54' \ 28'' \end{cases}$

5. Given two sides of a triangle equal to 77.245 and 92.341, and the angle opposite the first side equal to $124° 31' 48''$; to solve the triangle.

Ans. The question is impossible.

6. Given two sides of a triangle equal to 75.486 and 92.341, and the angle opposite the first side equal to $55° 28' 12''$; to solve the triangle.

Ans. The question is impossible.

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81. *Theorem. The sum of two sides of a triangle is to their difference, as the tangent of half the sum of the opposite angles is to the tangent of half their difference.* [B. p. 13.]

Proof. We have (fig. 1),

$$a : b = \sin. A : \sin. B;$$

whence, by the theory of proportions,

$$a + b : a - b = \sin. A + \sin. B : \sin. A - \sin. B,$$

which, expressed fractionally, is

$$\frac{a+b}{a-b} = \frac{\sin. A + \sin. B}{\sin. A - \sin. B}.$$

But, by (47),

$$\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{\text{tang.} \frac{1}{2}(A+B)}{\text{tang.} \frac{1}{2}(A-B)};$$

whence

$$\frac{a+b}{a-b} = \frac{\text{tang.} \frac{1}{2}(A+B)}{\text{tang.} \frac{1}{2}(A-B)}; \tag{136}$$

or

$$a + b : a - b = \text{tang.} \tfrac{1}{2}(A+B) : \text{tang.} \tfrac{1}{2}(A-B).$$

82. *Problem. To solve a triangle, when two of its sides and the included angle are given.* [B. p. 43.]

Solution. Let the two sides a and b (fig. 1) be given, and the included angle C; to solve the triangle.

First. To find the other two angles. Subtract the given angle C from 180°, and the remainder is the sum of A and B, for the sum of the three angles of a triangle is 180°; that is,

$$A + B = 180° - C,$$

and

$$\tfrac{1}{2}(A+B) = 90° - \tfrac{1}{2} C = \text{complement of } \tfrac{1}{2} C.$$

The difference of A and B is then found by (136)

$$a + b : a - b = \text{tang.} \tfrac{1}{2}(A+B) : \text{tang.} \tfrac{1}{2}(A-B).$$

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But we have

$$\text{tang. } \tfrac{1}{2}(A+B) = \text{cotan. } \tfrac{1}{2}C;$$

whence

$$\text{tang. } \tfrac{1}{2}(A-B) = \frac{a-b}{a+b} \text{ tang. } \tfrac{1}{2}(A+B) = \frac{a-b}{a+b} \text{ cotan. } \tfrac{1}{2}C. \quad (137)$$

The greater angle, which must be opposite the greater side, is then found by adding their half sum to their half difference; and the smaller angle by subtracting the half difference from the half sum.

Secondly. The third side is found by the proportion

$$\text{sin. } A : \text{sin. } C = a : c;$$

whence

$$c = \frac{a \text{ sin. } C}{\text{sin. } A}.$$

83. EXAMPLES.

1. Given two sides of a triangle equal to 99.341 and 1.234, and their included angle equal to 169° 58′; to solve the triangle.

Solution. Making $a = 99.341$, $b = 1.234$; and

$$C = 169° \, 58', \tfrac{1}{2} C = 84° \, 59';$$

we have, by (137),

$a + b =$	100.575	(ar. co.) 7.99751
$a - b =$	98.107	1.99170
$\frac{1}{2}(A+B) =$	5° 1′	tang. 8.94340
$\frac{1}{2}(A-B) =$	4° 53′ 39″	tang. 8.93261
$A =$	9° 54′ 39″	
$B =$	0° 7′ 21″	
$a =$	99.341	1.99712
$C =$	169° 58′	sin. 9.24110
$A =$	9° 54′ 39″	cosec. 10.76419
$c =$	100.56	2.00241

Ans. The third side = 100.56

The other angles $= \begin{cases} 9° \, 54' \, 39'' \\ 0° \;\; 7' \, 21'' \end{cases}$

2. Given two sides of a triangle equal to 10.121 and 15.421, and the included angle equal to $41° 2'$; to solve the triangle.

Ans. The other side $=$ 10.236

$$\text{The other angles} = \begin{cases} 98° \; 29' \; 32'' \\ 40° \; 28' \; 28'' \end{cases}$$

84. *Theorem. One side of a triangle is to the sum of the other two, as their difference is to the difference of the segments of the first side made by a perpendicular from the opposite vertex, if the perpendicular fall within the triangle; or to the sum of the distances of the extremities of the base from the foot of the perpendicular, if it fall without the triangle.* [B. p. 14.]

Proof. Let AC (figs. 12 and 13) be the side of triangle ABC on which the perpendicular is let fall, and BP the perpendicular.

From B as a centre with a radius equal to BC, the shorter of the other two sides, describe the circumference $CC'\,E'E$. Produce AB to E' and AC to C', if necessary.

Then, since AC and AB are secants, we have,

$$AC : AE' = AE : AC'.$$

But

$$AE' = AB + BE' = AB + BC$$

$$AE = AB - BE = AB - BC,$$

and

$$\text{(fig. 12) } AC' = AP - PC' = AP - PC$$

$$\text{(fig. 13) } AC' = AP + PC' = AP + PC;$$

whence

$$\text{(fig. 12) } AC : AB + BC = AB - BC : AP - PC$$

$$\text{(fig. 13) } AC : AB + BC = AB - BC : AP + PC.$$

85. *Problem. To solve a triangle, when its three sides are given.* [B. p. 43.]

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Solution. On the side b (figs. 2 and 3) let fall the perpendicular BP.

Then, by § 84,

$$(\text{fig. } 2)\ b : c + a = c - a : PA - PC$$

$$(\text{fig. } 3)\ b : c + a = c - a : PA + PC.$$

These proportions give the difference of the segments (fig. 2), or their sum (fig. 3). Then, adding the half difference to the half sum, we obtain the larger segment corresponding to the larger of the two sides a and c. And, subtracting the half difference from the half sum, we obtain the smaller segment.

Then, in triangles BCP and ABP, we have, by (4) and (99),

$$\cos. A = \frac{AP}{c};$$

and

$$(\text{fig. } 2)\ \cos. C = \frac{PC}{a},$$

$$(\text{fig. } 3)\ \cos. C = -\cos. BCP = -\frac{PC}{a}.$$

The third angle B is found by subtracting the sum of A and C from 180°.

86. *Corollary.* From the preceding section, we have

$$(\text{fig. } 2)\ PA - PC = \frac{(c + a)\ (c - a)}{b} = \frac{c^2 - a^2}{b}$$

$$(\text{fig. } 3)\ PA + PC = \frac{(c + a)\ (c - a)}{b} = \frac{c^2 - a^2}{b};$$

which, added to

$$(\text{fig. } 2)\ PA + PC = AC = b$$

$$(\text{fig. } 3)\ PA - PC = AC = b,$$

gives

$$2\ PA = \frac{c^2 - a^2}{b} + b = \frac{b^2 + c^2 - a^2}{b}.$$

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Hence

$$PA = \frac{b^2 + c^2 - a^2}{2b}$$

and

$$\cos. A = \frac{PA}{c} = \frac{b^2 + c^2 - a^2}{2bc}. \qquad (138)$$

87. *Corollary.* If (138) is cleared from fractions it becomes by transposition

$$a^2 = b^2 + c^2 - 2bc \cos. A. \qquad (139)$$

88. *Corollary.* Add unity to both sides of (138), and we have

$$1 + \cos. A = \frac{b^2 + c^2 - a^2}{2bc} + 1 = \frac{b^2 + 2bc + c^2 - a^2}{2bc}$$

$$= \frac{(b + c)^2 - a^2}{2bc}. \qquad (140)$$

Since the numerator of (140) is the difference of two squares, it may be separated into two factors, and we have

$$1 + \cos. A = \frac{(b + c + a)(b + c - a)}{2bc}.$$

Now, representing half the sum of the three sides of a triangle by s, we have

$$2s = a + b + c, \qquad (141)$$

and

$$2s - 2a = 2(s - a) = a + b + c - 2a = b + c - a. \quad (142)$$

If we substitute these values in the above equation, it becomes

$$1 + \cos. A = \frac{4s(s - a)}{2bc} = \frac{2s(s - a)}{bc}. \qquad (143)$$

But, by (55),

$$1 + \cos. A = 2(\cos. \tfrac{1}{2} A)^2.$$

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Hence

$$2\,(\cos.\tfrac{1}{2}A)^2 = \frac{2\,s\,(s-a)}{b\,c}$$

or

$$(\cos.\tfrac{1}{2}A)^2 = \frac{s\,(s-a)}{b\,c} \qquad (144)$$

$$\cos.\tfrac{1}{2}A = \sqrt{\left(\frac{s\,(s-a)}{b\,c}\right)}, \qquad (145)$$

which corresponds to proposition LXI. of B., p. 14.

In the same way, we have

$$\cos.\tfrac{1}{2}B = \sqrt{\left(\frac{s\,(s-b)}{a\,c}\right)} \qquad (146)$$

$$\cos.\tfrac{1}{2}C = \sqrt{\left(\frac{s\,(s-c)}{a\,b}\right)}. \qquad (147)$$

89. *Corollary.* Subtract both sides of (138) from unity, and we have

$$1-\cos.A = 1 - \frac{b^2+c^2-a^2}{2\,b\,c} = \frac{a^2+2\,b\,c-b^2-c^2}{2\,b\,c}$$

$$= \frac{a^2-(b-c)^2}{2\,b\,c}. \qquad (148)$$

Since the numerator of (148) is the difference of two squares, it may be separated into two factors, as follows,

$$1-\cos.A = \frac{(a-b+c)\,(a+b-c)}{2\,b\,c}.$$

But from (141)

$$2\,s-2\,b = 2\,(s-b) = a+b+c-2\,b = a-b+c \qquad (149)$$

$$2\,s-2\,c = 2\,(s-c) = a+b+c-2\,c = a+b-c. \qquad (150)$$

If we substitute these values in the above equations, it becomes

$$1-\cos.A = \frac{4\,(s-b)\,(s-c)}{2\,b\,c} = \frac{2\,(s-b)\,(s-c)}{b\,c}. \qquad (151)$$

But by (56),

$$1-\cos.A = 2\,(\sin.\tfrac{1}{2}A)^2.$$

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Hence, by reduction,

$$\sin.\tfrac{1}{2}A = \sqrt{\left(\frac{(s-b)\ (s-c)}{b\ c}\right)}. \tag{152}$$

In the same way, we have

$$\sin.\tfrac{1}{2}B = \sqrt{\left(\frac{(s-a)\ (s-c)}{a\ c}\right)} \tag{153}$$

$$\sin.\tfrac{1}{2}C = \sqrt{\left(\frac{(s-a)\ (s-b)}{a\ b}\right)}. \tag{154}$$

90. *Corollary.* The quotients of (152, 153, and 154) divided by (145, 146, and 147), are by (7)

$$\text{tang.}\tfrac{1}{2}A = \sqrt{\left(\frac{(s-b)\ (s-c)}{s\ (s-a)}\right)} \tag{155}$$

$$\text{tang.}\tfrac{1}{2}B = \sqrt{\left(\frac{(s-a)\ (s-c)}{s\ (s-b)}\right)} \tag{156}$$

$$\text{tang.}\tfrac{1}{2}C = \sqrt{\left(\frac{(s-a)\ (s-b)}{s\ (s-c)}\right)}. \tag{157}$$

91. The product of (143) by (151) is

$$1-(\cos.A)^2 = \frac{4\ s\ (s-a)\ (s-b)\ (s-c)}{b^2\ c^2}.$$

But from (9)

$$1-(\cos.A)^2 = (\sin.A)^2.$$

Hence

$$(\sin.A)^2 = \frac{4\ s\ (s-a)\ (s-b)\ (s-c)}{b^2\ c^2},$$

or

$$\sin.A = \frac{2\sqrt{[s\ (s-a)\ (s-b)\ (s-c)]}}{b\ c}. \tag{158}$$

92. *Scholium.* The problem would be impossible, if the given value of either side exceeded the sum of the other two.

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93. Examples.

1. Given the three sides of a triangle equal to 12.348, 13.561, and 14.091; to solve the triangle.

Solution. First Method.

Make (fig. 2) $a = 12.348$ $b = 13.561$
$c = 14.091$.

Then by § 91

$b = 13.561$	(ar. co.) 8.86771		
$c + a = 26.439$	1.42224		
$c - a = 1.743$	0.24130		
$PA - PC = 3.3982$	0.53125		
$PA = 8.4796$	0.92838		
$PC = 5.0814$			0.70598
$c = 14.091$	(ar. co.) 8.85106		
$a = 12.348$		(ar. co.)	8.90840
$A = 53° \ 0'$	cos. 9.77944		
$C = 65° \ 42'$		cos.	9.61438

$B = 180° - (A + C)$
$= 180° - 118° \ 42' = 61° \ 18'$.

Second Method.

By (145, 146, and 147),

$a = 12.348$		(ar. co.) 8.90840	(ar. co.) 8.90840
$b = 13.561$	(ar. co.) 8.86771		(ar. co.) 8.86771
$c = 14.091$	(ar. co.) 8.85106	(ar. co.) 8.85106	
$s = 20.000$	1.30103	1.30103	1.30103
$s - a = 7.652$	0.88377		
$s - b = 6.439$		0.80882	
$s - c = 5.909$			0.77151
	2 | 19.90357	2 | 19.86931	2 | 19.84865
cos.	9.95179	9.93466	9.92433

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$$\tfrac{1}{2}A = 26^\circ\ 30',\ \tfrac{1}{2}B = 30^\circ\ 39',\ \tfrac{1}{2}C = 32^\circ\ 51'$$

$$A = 53^\circ\ \ 0',\quad B = 61^\circ\ 18',\quad C = 65^\circ\ 42'.$$

$$\textit{Ans.}\quad \text{The angles} = \begin{cases} 53^\circ\ \ 0' \\ 61^\circ\ 18' \\ 65^\circ\ 42'. \end{cases}$$

In the same way equations (152–154) would furnish a third method, (155–157) a fourth method, and (158) a fifth method.

2. Given the three sides of a triangle equal to 17.856, 13.349, and 11.111; to solve the triangle.

$$\textit{Ans.}\quad \text{The angles} = \begin{cases} 93^\circ\ 19'\ 16'' \\ 48^\circ\ 16'\ 24'' \\ 38^\circ\ 24'\ 20''. \end{cases}$$

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CHAPTER VII.

LOGARITHMIC AND TRIGONOMETRICAL SERIES.

94. *Problem.* To develop the expression

$$(1+i)^{\frac{x}{i}} \tag{159}$$

in which i is an infinitesimal, into a series arranged according to powers of x.

Solution. The binomial theorem gives at once

$$(1+i)^{\frac{x}{i}} = 1 + \frac{x}{i}\,i + \frac{x}{i}\left(\frac{x}{i}-1\right)\frac{i^2}{1.2}$$

$$+\frac{x}{i}\left(\frac{x}{i}-1\right)\left(\frac{x}{i}-2\right)\frac{i^3}{1.2.3} + \&c. \tag{160}$$

But $\frac{x}{i}$ is infinite, and gives, therefore,

$$\frac{x}{i}-1=\frac{x}{i}, \quad \frac{x}{i}-2=\frac{x}{i}, \quad \&c. \tag{161}$$

which, substituted in (130), give

$$(1+i)^{\frac{x}{i}} = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \&c. \tag{162}$$

95. *Corollary.* When $x = 1$

(162) becomes

$$(1+i)^{\frac{1}{i}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c. \tag{163}$$

which we may denote by e.

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This quantity e is one of frequent occurrence in analysis, and is celebrated on account of its having been adopted by Napier as the base of his system of logarithms, which were called by him *hyperbolic logarithms*, but are known as the *Naperian logarithms*.

The value of e is easily computed, from the consideration that it is the *sum of the series* (163), *the first term of which is unity, and each succeeding term is obtained by dividing the preceding term by the number of the place of this preceding term.*

Thus

$$\begin{array}{rl}
1) & 1.000000 \\
2) & 1.000000 \\
3) & .500000 \\
4) & .166667 \\
5) & .041667 \\
6) & .008333 \\
7) & .001389 \\
8) & .000198 \\
9) & .000025 \\
 & .000003 \\
\hline
\end{array}$$

$$(1+i)^{\frac{1}{i}} = e = 2.71828 \tag{164}$$

The sixth place is neglected, in the sum of the decimals, as being inaccurate.

96. *Corollary.* The xth power of e is by (164 and 162)

$$e^x = (1+i)^{\frac{x}{i}} = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c. \tag{165}$$

97. *Corollary.* The i^{th} power of e is

$$e^i = 1 + i. \tag{166}$$

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98. *Corollary.* The logarithm of (166) is

$$\log. (1 + i) = i \log. e, \tag{167}$$

which gives, by reversing the sign of i,

$$\log. (1 - i) = - i \log. e. \tag{168}$$

99. *Problem.* To develop $\log. (1 - x)$ into a series of terms arranged according to the powers of x.

Solution. Let the series be denoted, as follows,

$$\log. (1 - x) = A + A_1 x + A_2 x^2 + \&c. \ldots + A_n x^n + \&c. \tag{169}$$

so that the number below the letter denotes the power of x of which the letter is the coefficient.

First. To find the value of A; let

$$x = 0$$

which reduces (169) to

$$\log. 1 = A = 0. \tag{170}$$

Secondly. To find the value of A_1; let

$$x = i$$

so that all the powers of x except the first can be neglected in the second member of (169), and the equations (168, 169, and 170,) give

$$\log. (1 - i) = A_1 i = - i \log. e \tag{171}$$

$$A_1 = - \log. e. \tag{172}$$

Thirdly. To find the value of A_n; let r, r', r'', &c. be the roots of the equation

$$x^n = 1, \text{ or } x^n - 1 = 0, \tag{173}$$

and by the theory of equations, we have for all values of x

$$x^n - 1 = (x - r)(x - r')(x - r'') \,\&c. \tag{174}$$

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Moreover the product of the negatives of the roots of an equation is equal to the constant term, which is, in this case, — 1, that is,

$$-1 = (-r)\,(-r')\,(-r'')\ \&c. \tag{175}$$

The quotient of (174) by (175) is

$$1 - x^n = \frac{x-r}{-r} \cdot \frac{x-r'}{-r'} \cdot \frac{x-r''}{-r''} \cdot \&c.$$

$$= \left(1 - \frac{x}{r}\right)\left(1 - \frac{x}{r'}\right)\left(1 - \frac{x}{r''}\right) \&c. \tag{176}$$

the logarithm of which is

$$\log.\,(1 - x^n) = \log.\left(1 - \frac{x}{r}\right) + \log.\left(1 - \frac{x}{r'}\right)$$

$$+ \log.\left(1 - \frac{x}{r''}\right) + \&c. \tag{177}$$

But by substituting x^n for x in (169), and the values of A and A_1 we have

$$\log.\,(1 - x^n) = -\log.\,e\,x^n + A_2\,x^{2n} + \&c. \tag{178}$$

and any term of the second member of (177), as the first, is by (169)

$$\log.\left(1 - \frac{x}{r}\right) = -\log.\,e\,\frac{x}{r} + \&c. \ldots + A_n\,\frac{x^n}{r^n}. \tag{179}$$

Since r is a root of the equation (173), that is, since

$$r^n = 1, \tag{180}$$

the term of (179), multiplied by x^n, becomes $A_n\,x^n$, which is independent of the particular root r, r', &c., and, therefore, the same for each term of the second member of (177). The sum of all the terms of the second member of (177), which are multiplied by x^n, is equal to either of them multiplied by their number, which is n, that is, it is

$$n\,A_n\,x^n. \tag{181}$$

Hence this term must be equal to the term of (178), which is multiplied by x^n, or

$$n\,A_n\,x^n = -\log.\,e\,x^n \tag{182}$$

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$$A_n = -\frac{1}{n} \log. e, \tag{183}$$

and the resulting value of (169) is

$$\log. (1 - x) = \log. e\,(- x - \tfrac{1}{2} x^2 - \tfrac{1}{3} x^3 - \tfrac{1}{4} x^4 - \&c.) \tag{184}$$

100. *Corollary.* By reversing the sign of x (184) becomes

$$\log. (1 + x) = \log. e\,(x - \tfrac{1}{2} x^2 + \tfrac{1}{3} x^3 - \tfrac{1}{4} x^4 + \tfrac{1}{5} x^5 - \&c.) \tag{185}$$

101. *Corollary.* The remainder of (185) diminished by (184) is

$$\log. \frac{1 + x}{1 - x} = 2 \log. e\,(x + \tfrac{1}{3} x^3 + \tfrac{1}{5} x^5 + \tfrac{1}{7} x^7 + \&c.) \tag{186}$$

102. *Corollary.* Since

$$a + x = a\left(1 + \frac{x}{a}\right)$$

we have by (185)

$$\log. (a + x) = \log. a + \log. \left(1 + \frac{x}{a}\right)$$

$$= \log. a + \log. e\left(\frac{x}{a} - \tfrac{1}{2}\frac{x^2}{a^2} + \tfrac{1}{3}\frac{x^3}{a^3} - \tfrac{1}{4}\frac{x^4}{a^4} + \&c.\right) \tag{187}$$

103. *Corollary.* Equations (184), (185) and (186) may be used in calculating logarithmic tables. But, for this purpose, log. e must first be obtained, which is very easy, since it is, by the definition of logarithms, the root of the equation

$$10^x = e = 2.71828$$

which gives

$$\log. e = x = 0.43429. \tag{188}$$

104. Examples.

1. Find the logarithm of 1.1.

Solution. By making in (185),

$$x = 0.1$$

we have

$$\tfrac{1}{2} x^2 = 0.005$$
$$\tfrac{1}{3} x^3 = 0.000333$$
$$\tfrac{1}{4} x^4 = 0.000025$$
$$\tfrac{1}{5} x^5 = 0.000002$$

whence $x - \tfrac{1}{2} x^2 + \tfrac{1}{3} x^3 - \tfrac{1}{5} x^5 = 0.09536$

$\log. (1 + x) = (0.09536)\ (\log. e) = 0.09536 \times 0.43429 = 0.04139$

2. Find the logarithm of 625, knowing that

$$\log. 624 = 2.79518.$$

Solution. In this case we have in (187)

$$a = 624,\ x = 1,\ \frac{x}{a} = \frac{1}{624}$$

and $\frac{x}{a}$ is so small, that its square and higher powers may be neglected in (187), whence

$$\log. 625 = \log. 624 + \frac{\log. e}{624}$$

$$= 2.79518 + \frac{0.43429}{624} = 2.79518 + 0.00070$$

$$= 2.79588.$$

3. Find the logarithm of .9. *Ans.* -0.04576 or $\bar{9}.95424$.

4. Find the logarithm of 1.01. *Ans.* 0.00432.

5. Find the logarithm of 1.095. *Ans.* 0.03941.

6. Find the logarithm of 1.003. *Ans.* 0.00130.

7. Find the logarithm of 463, knowing that

$$\log. 462 = 2.66464.$$

Ans. 2.66558.

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8. Find the logarithm of 1291, knowing that

$$\log. 1290 = 3.11059.$$

Ans. 3.11093.

9. Find the logarithm of 123.6, knowing that

$$\log. 123 = 2.08991$$

Ans. 2.09202.

105. *Problem. To express sines and cosines by means of exponential functions.*

Solution. The first member of the equation

$$\cos.^2 x + \sin.^2 = 1 \tag{189}$$

may be written $\cos.^2 x - (-\sin.^2 x)$, that is the difference of the two squares $\cos.^2 x$ and $(-\sin.^2 x)$, of which the roots are $\cos. x$ and $\sin. x \sqrt{-1}$. This first member is, therefore, the product of the sum and difference of these two roots, or (189) may be written

$$(\cos. x + \sin. x . \sqrt{-1})\ (\cos. x - \sin. x . \sqrt{-1}) = 1.$$

The logarithm of this equation is

$$\log. (\cos. x + \sin. x . \sqrt{-1}) + \log. (\cos. x - \sin. x . \sqrt{-1}) = 0$$

or

$$\log. (\cos. x + \sin. x . \sqrt{-1}) = -\log. (\cos. x - \sin. x . \sqrt{-1}). \tag{190}$$

Denote either member of 190 by y, so that

$$\log. (\cos. x + \sin. x . \sqrt{-1}) = y,$$

$$\log. (\cos. x - \sin. x . \sqrt{-1}) = -y \tag{191}$$

or

$$\cos. x + \sin. x . \sqrt{-1} = 10^y,\ \cos. x - \sin. x . \sqrt{-1} = 10^{-y}. \tag{192}$$

The sum of the two last equations is

$$2 \cos. x = 10^y + 10^{-y}. \tag{193}$$

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Hence by (55 and 56)

$$\cos.^2 \tfrac{1}{2} x = \tfrac{1}{2} (1 + \cos. x) = \tfrac{1}{4} (2 + 2 \cos. x) = \tfrac{1}{4} (10^y + 2 + 10^{-y})$$

$$-\sin.^2 \tfrac{1}{2} x = \tfrac{1}{2} (\cos. x - 1) = \tfrac{1}{4} (2 \cos. x - 2) = \tfrac{1}{4} (10^y - 2 + 10^{-y})$$

of which the square roots are

$$\cos. \tfrac{1}{2} x = \tfrac{1}{2} (10^{\frac{1}{2} y} + 10^{-\frac{1}{2} y})$$

$$\sin. \tfrac{1}{2} x \sqrt{-1} = \tfrac{1}{2} (10^{\frac{1}{2} y} - 10^{-\frac{1}{2} y})$$

and the sum of these two equations is

$$\cos. \tfrac{1}{2} x + \sin. \tfrac{1}{2} x . \sqrt{-1} = 10^{\frac{1}{2} y}. \qquad (194)$$

The comparison of (194) with the first equation of (192) shows that x may be changed into $\frac{1}{2} x$, provided that y is changed into $\frac{1}{2} y$. The same changes may, therefore, also be made in (194), or $\frac{1}{2} x$ may be changed into its half, that is, into $\frac{1}{4} x$, provided $\frac{1}{2} y$ is changed into $\frac{1}{4} y$, which gives

$$\cos. \tfrac{1}{4} x + \sin. \tfrac{1}{4} x . \sqrt{-1} = 10^{\frac{1}{4} y}. \qquad (195)$$

A repetition of this change gives

$$\cos. \tfrac{1}{8} x + \sin. \tfrac{1}{8} x . \sqrt{-1} = 10^{\frac{1}{8} y}. \qquad (196)$$

By continuing this process, x may be divided by any power of 2, however great, provided y is divided by the same power. Let, then,

$$m = 2^n \qquad (197)$$

and we have

$$\cos. \frac{x}{m} + \sin. \frac{x}{m} . \sqrt{-1} = 10^{\frac{y}{m}}. \qquad (198)$$

The logarithm of which is

$$\log. \left(\cos. \frac{x}{m} + \sin. \frac{x}{m} \sqrt{-1}\right) = \frac{y}{m}. \qquad (199)$$

But if in (197) n is made infinite, m will also be infinite, and $\frac{x}{m}$ will

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be an infinitesimal, whose cosine is unity and sine equal to its arc, that is, (199) becomes

$$\log.\ (1 + \frac{x}{m}\sqrt{-1}) = \frac{y}{m}. \qquad (200)$$

But, again, since $\frac{x}{m}$ is an infinitesimal, (200) becomes by means of (167)

$$\log.\ e\ .\ \frac{x}{m}\sqrt{-1} = \frac{y}{m},\ \text{or}\ y = x\sqrt{-1}\ \log.\ e, \qquad (201)$$

which substituted in (191) gives

$$\log.\ (\cos.\ x + \sin.\ x\ .\ \sqrt{-1}) = x\sqrt{-1}\ .\ \log.\ e = \log.\ e^{x\sqrt{-1}}$$

$$\log.\ (\cos.\ x - \sin.\ x\ .\ \sqrt{-1}) = -x\sqrt{-1}\log.\ e = \log.\ e^{-x\sqrt{-1}} \qquad (202)$$

or

$$\cos.\ x + \sin.\ x\ .\ \sqrt{-1} = e^{x\sqrt{-1}}$$

$$\cos.\ x - \sin.\ x\ .\ \sqrt{-1} = e^{-x\sqrt{-1}}. \qquad (203)$$

106. *Corollary.* Half the sum of (203) is

$$\cos.\ x = \tfrac{1}{2}\,(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}), \qquad (204)$$

and half their difference, multiplied by $\sqrt{-1}$, is

$$\sin.\ x = -\tfrac{1}{2}\,(e^{x\sqrt{-1}} - e^{-x\sqrt{-1}})\sqrt{-1}. \qquad (205)$$

107. *Problem. To develop cos. x and sin. x in sines arranged according to powers of x.*

Solution. Since we have

$$(x\sqrt{-1})^2 = -x^2, (x\sqrt{-1})^3 = -x^3\sqrt{-1}, (x\sqrt{-1})^4 = x^4, \&c. \qquad (206)$$

the substitution of $x\sqrt{-1}$ for x in (165) gives

$$e^{x\sqrt{-1}} = 1 + x\sqrt{-1} - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3}$$

$$+ \frac{x^4}{1.2.3.4} + \frac{x^5\sqrt{-1}}{1.2.3.4.5} - \&c. \qquad (207)$$

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which gives by reversing the sign of x

$$e^{-x\sqrt{-1}} = 1 - x\sqrt{-1} - \frac{x^2}{1.2} + \frac{x^3\sqrt{-1}}{1.2.3}$$

$$+ \frac{x^4}{1.2.3.4} - \frac{x^5\sqrt{-1}}{1.2.3.4.5} - \&c. \qquad (208)$$

Half the sum of (207) and (208) is, by (204),

$$\cos. x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \&c. \qquad (209)$$

Half their difference, multiplied by $\sqrt{-1}$, is by (205)

$$\sin. x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c. \qquad (210)$$

which are the series required. But it must not be forgotten that, in the second member of these equations, x is expressed in terms of the radius as unity.

108. *Corollary.* Equations (209) and (210) can be used for calculating tables of sines and cosines.

109. Examples.

1. Find the sine and cosine of $13° 25'$.

Solution. In this case $x = (13° 25') \sin. 1' = 805 \sin. 1'$, or by (13)

$$x = 805 \times 0.0002908 = .234164$$

$$\frac{x^2}{1.2} = 0.027419, \qquad \frac{x^3}{1.2.3} = 0.002140$$

$$\frac{x^4}{1.2.3.4} = 0.000125, \qquad \frac{x^5}{1.2.3.4.5} = 0.000006$$

Hence $\cos. x = 0.\ 97271$ $\qquad \sin. x = 0.23203$

2. Find the sine and cosine of $6° 10'$.

Ans. $\sin. 6° 10' = 0.10742$

$\cos. 6° 10' = 0.99421$

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3. Find the sine and cosine of $3°$.

Ans. $\sin. 3° = 0.05234$

$\cos. 3° = 0.99863$

4. Find the sine and cosine of $10° 38'$.

Ans. $\sin. 10° 38' = 0.18452$

$\cos. 10° 38' = 0.98283$

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NAVIGATION AND SURVEYING.

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NAVIGATION AND SURVEYING.

CHAPTER I.

PLANE SAILING.

1. The daily revolution of the earth is performed around a straight line, passing through its centre, which is called the *earth's axis.*

The extremities of this axis on the surface of the earth are the *terrestrial poles*, one being the *north pole*, and the other the *south pole.*

The section of the earth, made by a plane passing through its centre and perpendicular to its axis, is the *terrestrial equator.* [B. p. 48.]

2. *Parallels of latitude* are the circumferences of small circles, the planes of which are parallel to the equator.

3. *Meridians* are the circumferences of great circles, which pass from one pole to the other.

The *first meridian* is one arbitrarily assumed, to which all others are referred. In most countries, that has been taken as the first meridian which passes through the capital of the country.

4. The *latitude* of a place is its angular distance from the equator, the vertex of the angle being at the centre of the

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earth; or, it is the arc of the meridian, passing through the place, which is comprehended between the place and the equator. [B. p. 48.]

Latitude is reckoned north and south of the equator from 0° to 90°.

5. The *difference of latitude* of two places is the angular distance between the parallels of latitude in which they are respectively situated, the vertex of the angle being at the centre of the earth; or it is the arc of a meridian which is comprehended between the parallels of latitude. [B. p. 52.]

The difference of latitude of two places is equal to the difference of their latitudes, if they are on the same side of the equator; and to the sum of their latitudes, if they are on opposite sides of the equator. B. p. 50.]

6. The *longitude* of a place is the angle made by the plane of the first meridian with the plane of the meridian passing through the place; or it is the arc of the equator comprehended between these two meridians. [B. p. 48.]

Longitude is reckoned East and West of the first meridian from 0° to 180°; or it may be reckoned towards the west from 0° to 360°.

7. The *difference of longitude* of two places is the angle contained between the planes of the meridians passing through the two places; or it is the arc of the equator comprehended between these two meridians.

The difference of longitude of two places is equal to the difference of their longitudes, if they are on the same side of the first meridian; and to the sum of their longitudes, if they are on opposite sides of the first meridian, unless their sum be greater than 180°; *in which case the sum must be subtracted from* 360° *to give the difference of longitude.* [B. p. 50.]

8. The *distance* between two places in Navigation is the portion of a curve which would be described by a ship sailing from one place to the other in a path, which crosses every meridian at the same angle. [B. p. 52.]

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9. The *course* of the ship, or the *bearing* of the two places from each other, is the angle which the ship's path makes with the meridian. [B. p. 52.]

10. The *departure* of two places is the distance of either from the meridian of the other, when they are so near each other that the earth's surface may be considered as plane and its curvature neglected. But, if the two places are at a great distance from each other, the distance is to be divided into small portions, and the *departure* of the two places is the sum of the departures corresponding to all these portions.

11. Instead of dividing the quadrant into 90 degrees, navigators are in the habit of dividing it into eight equal parts called *points;* and of subdividing the points into halves and quarters. A point, therefore, is equal to one eighth of 90°, or to 11° 15′. [B. p. 52.]

Names are given to the directions determined by the different points, as in the diagram (fig. 14), which represents the face of the card of the *Mariner's Compass.*

The *Mariner's Compass* consists of this card, attached to a magnetic needle, which has the property of constantly pointing toward the north, and thereby determining the ship's course.

On page 53 of the Navigator a table is given of the angles which every *point* of the compass makes with the meridian, and on page 169, table XXV. the log., sines, &c. are given.

12. The object of *Plane Sailing* is to calculate the Distance, Course, or Bearing, Difference of Latitude and Departure, when either two of them are known. [B. p. 52.]

13. *Problem. To find the difference of latitude and departure, when the distance and course are known.* [B. p. 54.]

Solution. First. When the distance is so small that the curvature of the earth's surface may be neglected. Let AB (fig. 15) be the

distance. Draw through A the meridian AC, and let fall on it the perpendicular BC. The angle A is the course, AC is the difference of latitude, and BC is the departure. Then, by (21 and 22),

$$\text{Diff. of lat.} = \text{dist.} \times \text{cos. course}, \qquad (211)$$

$$\text{Departure} = \text{dist.} \times \text{sin. course}. \qquad (212)$$

Secondly. When the distance is great, as AB (fig. 16), then divide it into smaller portions, as $A\,a$, $a\,b$, $b\,c$, &c. Through the points of division, draw the meridians AN, $a\,n$, $b\,p$, &c. Let fall the perpendiculars $a\,m$, $b\,n$, $c\,p$, &c. Then, as the course is every where the same, each of the angles $m\,A\,a$, $n\,a\,b$, $p\,b\,c$, &c. is equal to the angle A, or the course. Moreover, the distances, $A\,m$, $a\,n$, $b\,p$, &c. are the differences of latitude respectively of A and a, a and b, b and c, &c. Also $a\,m$, $b\,n$, $c\,p$, &c. are the departures of the points A and a, a and b, b and c, &c. Therefore, as the difference of latitude of A and B is evidently equal to the sum of these partial differences of latitude; and as the departure of A and B is by § 10 equal to the sum of the partial departures, we have

$$\text{Diff. of lat.} = A\,m + a\,n + b\,p + \text{\&c.}$$

$$\text{Departure} = a\,m + b\,n + c\,p + \text{\&c.}$$

But the right triangles $m\,A\,a$, $n\,a\,b$, $p\,b\,c$, &c. give by (211) and (212)

$$A\,m = A\,a \times \text{cos. course}, \quad a\,m = A\,a \times \text{sin. course};$$

$$a\,n = a\,b \times \text{cos. course}, \quad b\,n = a\,b \times \text{sin. course};$$

$$b\,p = b\,c \times \text{cos. course}, \quad c\,p = b\,c \times \text{sin. course}.$$

$$\text{\&c. \&c.}$$

The sums of these equations give

$$\begin{aligned} \text{Diff. of lat.} &= A\,m + a\,n + b\,p + \text{\&c.} \\ &= (A\,a + a\,b + b\,c + \text{\&c.}) \times \text{cos. course}, \\ \text{Departure} &= a\,m + b\,n + c\,p + \text{\&c.} \\ &= (A\,a + a\,b + b\,c + \text{\&c.}) \times \text{sin. course}. \end{aligned}$$

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But

$$A\,a + a\,b + b\,c + \&c. = AB = \text{distance.}$$

Hence,

$$\text{Diff. of lat.} = \text{dist.} \times \text{cos. course},$$

$$\text{Departure} = \text{dist.} \times \text{sin. course};$$

precisely the same with (211) and (212).

This shows that the method of calculating the difference of latitude and departure is the same for all distances, and that *all the problems of Plane Sailing may be solved by the right triangle* (fig. 15). [B. p. 52.]

Tables of difference of latitude and departure are given in pages 1–6, Tables I. and II. of the Navigator, which might be calculated by (211) and (212).

14. *Problem. To find the distance and difference of latitude, when the course and departure are known.* [B. p. 55.]

Solution. There are given (fig. 15) the angle A and the side BC. Hence, by (23) and (24),

$$\text{Distance} = \text{departure} \times \text{cosec. course}, \tag{213}$$

$$\text{Diff. of lat.} = \text{departure} \times \text{cotan. course}. \tag{214}$$

15. *Problem. To find the distance and departure, when the course and difference of latitude are known.* [B. p. 55.]

Solution. There are given (fig. 15) the angle A and the side AC. Then, by (25) and (26),

$$\text{Distance} = \text{diff. of lat.} \times \text{sec. course}, \tag{215}$$

$$\text{Departure} = \text{diff. of lat.} \times \text{tang. course}. \tag{216}$$

16. *Problem. To find the course and difference of latitude, when the distance and departure are known.* [B. p. 27.]

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Solution. There are given (fig. 15) the hypothenuse AB and the side BC. Then, by (27) and (29),

$$\text{sin. course} = \frac{\text{departure}}{\text{distance}}, \tag{217}$$

$$\text{Diff. of lat.} = \surd[(\text{dist.}^2) - (\text{departure})^2]. \tag{218}$$

17. *Problem. To find the course and departure, when the distance and difference of latitude are known.* [B. p. 56.]

Solution. There are given (fig. 15) the hypothenuse AB and the leg AC. Then, by (27) and (29),

$$\text{cos. course} = \frac{\text{diff. of lat.}}{\text{distance}}, \tag{219}$$

$$\text{Departure} = \surd[(\text{dist.})^2 - (\text{diff. of lat.})^2]. \tag{220}$$

18. *Problem. To find the course and distance, when the departure and difference of latitude are known.* [B. p. 57.]

Solution. There are given (fig. 15) the legs AC and BC. Then, by (30) and (32),

$$\text{tang. course} = \frac{\text{departure}}{\text{diff. of lat.}}, \tag{221}$$

$$\text{Dist.} = \text{diff. of lat.} \times \text{sec. course}. \tag{222}$$

19. Examples.

1. A ship sails from latitude 3° 45′ S., upon a course N. by E., a distance of 2345 miles; to find the latitude at which it arrives, and the departure which it makes.

Ans. Latitude = 34° 35′ N.
Departure = 458 miles.

2. A ship sails from latitude 62° 19° N., upon a couse W. N. W., till it makes a departure of 1000 miles; to find the latitude at which it arrives, and the distance sailed.

Ans. Latitude = 69° 13′ N.
Distance = 1082 miles.

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3. The bearing of Paris from Athens is N. 54° 56′ W.; find the distance and departure of these two places from each other.

Ans. Distance = 1135 miles.

Departure = 929 miles.

4. A ship sails from latitude 72° 3′ S., a distance of 2000 miles, upon a course between the north and the west, that is, *northwesterly*, until it makes a departure of 100 miles; find the latitude at which it arrives, and the course.

Ans. Latitude = 43° 11′ S.

Course = N. 30° W.

5. The distance from New Orleans to Portland is 1257 miles; find the bearing and departure.

Ans. Bearing = N. 49° 24′ E.

Departure = 954 miles.

6. The departure of Boston from Canton is 8790 miles; find the bearing and distance.

Ans. Bearing = N. 82° 31′ E.

Distance = 8865 miles.

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CHAPTER II.

TRAVERSE SAILING.

20. A TRAVERSE is an irregular track made by a ship when sailing on several different courses.

The object of *Traverse Sailing* is to reduce a traverse to a single course, where the distances sailed are so small that the earth's surface may be considered as a plane. [B. p. 59.]

21. *Problem. To reduce several successive tracks of a ship to one; that is, to find the single track, leading to the place, which the ship has actually reached, by sailing on a traverse.* [B. p. 59.]

Solution. Suppose the ship to start from the point A (fig. 17), and to sail, first from A to B, then from B to C, then from C to E, and lastly from E to F; to find the bearing and distance of F from A. Call the differences of latitude, corresponding to the 1st, 2d, 3d, and 4th tracks, the 1st, 2d, 3d, and 4th differences of latitude; and call the corresponding departures the 1st, 2d, 3d, and 4th departures. Then we need no demonstration to prove that

$$\text{Diff. of lat. of } A \text{ and } F = \text{1st diff. of lat.} - \text{2d diff. of lat.}$$
$$+ \text{3d diff. of lat.} - \text{4th diff. \&c.};$$

or that *the difference of latitude* of A and F *is found by taking the sum of the differences of latitude corresponding to the northerly courses, and also the sum of those corresponding to the southerly courses; and the difference of these sums is the required difference of latitude.*

By neglecting the earth's curvature, we also have,

$$\text{Dep. of } A \text{ and } F = \text{1st dep.} - \text{2d dep.} - \text{3d dep.} + \text{4th dep.}$$

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or *the departure of A and F is found by taking the sum of the departures corresponding to the easterly courses, also the sum of those corresponding to the westerly courses; and the difference of these sums is the required departure.*

Having thus found the difference of latitude and departure of A and F, their distance and bearing are found by § 18.

22. The calculations of traverse sailing are usually put into a tabular form, as in the following example. In the *first* column of the table are the numbers of the courses; in the *second* and *third* columns are the courses and distances; in the *fourth* and *fifth* columns are the differences of latitude, the column, headed N, corresponding to the northerly courses, and that headed S, to the southerly courses; in the *sixth* and *seventh* columns are the departures, the column headed E, corresponding to the easterly courses, and that, headed W, to the westerly courses. [B. p. 59.]

23. Examples.

1. A ship sails on several successive tracks, in the order and with the courses and distances of the first three columns of the following table; find the bearing and distance of the place at which the ship arrives, from that from which it started.

No.	Course.	Dist.	N.	S.	E.	W.
1	N. N. E.	30	27.7		11.5	
2	N. W.	80	56.6			56.6
3	West.	60				60.0
4	S. E. by S.	55		45.7	30.6	
5	North.	43	43.0			
6	S. by W.	152		149.1		29.7
	Sum of columns,		127.3	194.8	42.1	146.3
				127.3		42.1
			Diff. lat. =	67.5 S.	dep. =	104.2 W.

Dep.	= 104.2		2.01787		
Diff. of lat.	= 67.5	(ar. co.)	8.17070		1.82930
Bearing	= 57° 4′	tang.	0.18857	sec.	0.26467
Dist.	= 124.2				2.09397

Ans. Bearing = S. 57° 4′ W.

Distance = 124.2 miles.

2. A ship sails on the following successive tracks, South 10 miles, W. S. W. 25 miles, S. W. 30 miles, and West 20 miles; it is bound to a port which is at a distance of 100 miles from the place of starting, and its bearing is W. by S.

Required the bearing and distance of the port to which the ship is bound, from the place at which it has arrived.

Ans. Bearing = N. 57° 47′ W.

Distance = 40 miles.

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CHAPTER III.

PARALLEL SAILING.

24. Parallel Sailing considers only the case where the ship sails exactly east or west, and therefore remains constantly on the same parallel of latitude. Its object is to find the change in longitude corresponding to the ship's track. [B. p. 63.]

25. *Problem. To find the difference of longitude in parallel sailing.* [B. p. 65.]

Solution. Let AB (fig. 18) be the distance sailed by the ship on the parallel of latitude AB. As the course is exactly east or west, the distance sailed must be itself equal to its departure.

The latitude of the parallel is ADA' or AA'. The angle $AEB = A'DB'$, or the arc $A'B$, is the difference of longitude. Denote the radius of the earth $A'D = B'D = AD$ by R, and the radius of the parallel $AE = BE$ by r; also the circumference of the earth by C, and that of the parallel by c.

Since AB and $A'B'$ correspond to the equal angles AEB and $A'DB'$, they must be similar arcs, and give the proportion,

$$AB : A'B' = c : C,$$

or
$$\text{Dep.} : \text{diff. long.} = c : C.$$

But, as circumferences are proportional to their radii,

$$c : C = r : R.$$

Hence, leaving out the common ratio,

$$\text{Dep.} : \text{diff. long.} = r : R.$$

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Putting the product of the extremes equal to that of the means,

$$r \,.\, \text{diff. of long.} = R \,.\, \text{departure.}$$

But, in the triangle $A\,D\,E$, since

$$D\,A\,E = A\,D\,A' = \text{latitude},$$

we have, from (22),

$$r = R \times \text{cos. lat.}$$

which, substituted in the above equation, gives, if the result is divided by R,

$$\text{Diff. long.} \times \text{cos. lat.} = \text{departure.} \tag{223}$$

Hence, by (8),

$$\text{Diff. long.} = \frac{\text{departure}}{\text{cos. lat.}} = \text{dep.} \times \text{sec. lat.} \tag{224}$$

26. *Corollary.* Since the distance is the same as the departure in parallel sailing, the word *distance* may be substituted for *departure* in (223) and (224).

27. *Corollary.* It appears, from (223) and (224,) that if a right triangle (fig. 18) is constructed, the hypothenuse of which is the difference of longitude, and one of the acute angles the latitude, the leg adjacent to this angle is the departure. *All the cases of parallel sailing may, then, be reduced to the solution of this triangle.*

28. *Problem. To find the distance between two places which are upon the same parallel of latitude.*

Solution. This problem is solved at once by (223).

29. The Table, p. 64, of the Navigator, which "shows for every degree of latitude how many miles distant two meridians are, whose difference of longitude is one degree," is readily calculated by this problem.

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30. Examples.

1. A ship sails from Boston 1000 miles exactly east; find the longitude at which it arrives.

Ans. Longitude sought $= 48° 32'$ W.

2. Find the distance of Barcelona (Spain) from Nantucket (Massachusetts).

Ans. Distance $= 3252$ miles.

3. Find the distance between two meridians, whose difference of longitude is one degree in the latitude of 45°.

Ans. Distance $= 42.43$ miles.

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CHAPTER IV.

MIDDLE LATITUDE SAILING.

31. THE object of *Middle Latitude Sailing* is to give an approximative method of calculating the difference of longitude, when the difference of latitude is small. [B. p. 66.]

32. *Problem. To find the difference of longitude by Middle Latitude Sailing, when the distance and course are known, and also the latitude of either extremity of the ship's track.* [B. p. 71.]

Solution. The difference of latitude and departure are found by (211) and (212),

$$\text{diff. lat.} = \text{dist.} \times \text{cos. course}$$

$$\text{departure} = \text{dist.} \times \text{sin. course.}$$

The difference of longitude may then be found by means of (224). But there is a difficulty with regard to the latitude to be used in (224); for, of the two extremities of the ship's track, the latitude of one is smaller, while the latitude of the other extremity is larger than the latitude of the rest of the track. Navigators have evaded this difficulty by using the *Middle Latitude* between the two, as sufficiently accurate, when the difference of latitude is small. Now the middle latitude is the arithmetical mean between the latitudes of the extremities, so that we have,

Middle lat. $= \frac{1}{2}$ sum of the lats. of the extremities of the track; (225)

and, by (224),

$$\text{diff. long.} = \frac{\text{departure}}{\text{cos. mid. lat.}} = \text{dep.} \times \text{sec. mid. lat.} \qquad (226)$$

or, by substituting (212),

$$\text{diff. long.} = \text{dist.} \times \text{sin. course} \times \text{sec. mid. lat.} \tag{227}$$

"This method of calculating the difference of longitude may be rendered perfectly accurate by applying to the middle latitude a correction," which is given in the Navigator, and the method of computing which will be explained in the succeeding chapter. [B. p. 76.]

33. By combining the triangle (fig. 15) of Plane sailing with that (fig. 18) of Parallel sailing, *a triangle* (fig. 19) *is obtained, by which all the cases of Middle Latitude sailing may be solved.*

34. *Problem. To find the distance and bearing of two places from each other, when their latitudes and longitudes are known.* [B. p. 68.]

Solution. From (fig. 19) we have

$$\text{departure} = \text{diff. long.} \times \text{cos. mid. lat.} \tag{228}$$

$$\text{tang. bearing} = \frac{\text{departure}}{\text{diff. lat.}} \tag{229}$$

$$\text{dist.} = \text{diff. lat.} \times \text{sec. bearing.} \tag{230}$$

35. *Problem. To find the course, distance, and difference of longitude, when both latitudes and the departure are given.* [B. p. 70.]

Solution. The difference of longitude is found by (226), the course by (229), and the distance by (230).

36. *Problem. To find the departure, distance, and difference of longitude, when both latitudes and the course are given.* [B. p. 72.]

Solution. The departure is found by the formula

$$\text{departure} = \text{diff. lat.} \times \text{tang. course}; \tag{231}$$

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the distance by (230); and the difference of longitude may be found by (226), or by substituting (231) in (226)

$$\text{diff. long.} = \text{diff. lat.} \times \text{tang. course} \times \text{sec. mid. lat.} \qquad (232)$$

37. *Problem. To find the course, departure, and difference of longitude, when both latitudes and the distance are given.* [B. p. 73.]

Solution. The course is found by the formula.

$$\text{cos. course} = \frac{\text{diff. lat.}}{\text{dist.}}; \qquad (233)$$

the departure by

$$\text{departure} = \text{dist.} \times \text{sin. course}; \qquad (234)$$

and the difference of longitude by (226).

38. *Problem. To find the difference of latitude, distance, and difference of longitude, when one latitude, course, and departure are given.* [B. p. 74.]

Solution. The difference of latitude is found by the formula

$$\text{diff. lat.} = \text{dep.} \times \text{cotan. course}; \qquad (235)$$

the distance by the formula

$$\text{dist.} = \text{dep.} \times \text{cosec. course}; \qquad (236)$$

and the difference of longitude by (226).

39. *Problem. To find the course, difference of latitude, and difference of longitude, when one latitude, the distance, and departure are given.* [B. p. 75.]

Solution. The course is found by the formula.

$$\text{sin. course} = \frac{\text{dep.}}{\text{dist.}}; \qquad (237)$$

the difference of latitude by the formula

$$\text{diff. lat.} = \text{dist.} \times \text{cos. course}; \qquad (238)$$

and the difference of longitude by (226).

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40. Examples.

1. A ship sailed from Halifax (Nova Scotia) a distance of 2509 miles, upon a course S. 79° 34′ E.; find the place at which it arrived.

Solution. By § 32,

dist.	= 2509		3.39950		3.39950
course	= 79° 34′	cos.	9.25790	sin.	9.99276
diff. lat. = 454′ =	7° 34′ S.		2.65740		
given lat.	= 44° 36′ N.	mid. lat.	= 40° 49′		
required lat.	= 37° 2′ N.	cor.	= 7′		
		cor. mid. lat.	= 40° 56′	sec.	10.12178
diff. long. = 3266′ =	54° 26′ E.				3.51404
given long.	= 63° 28′ W.				
required long.	= 9° 2′ W.				

Ans. The place arrived at is one mile south of Cape St. Vincent in Portugal.

2. Find the bearing and distance of Canton from Washington.

Solution. by § 34,

lat. of Washington	= 38° 53′ N.	long. =	77° 3′ W.
lat. of Canton	= 23° 7′ N.	long. =	113° 14′ E.
diff. lat.	= 946′ = 15° 46′,	sum of longs. =	190° 17′
mid. lat.	= 31° 0′	diff. long. =	169° 43′ = 10183′
cor.	= 31′		

cor. mid. lat.	= 31° 31′	cos.	9.93069		
diff. long.	= 10183′		4.00788		
diff. lat.	= 946′	ar. co.	7.02411		2.97589
bearing	= S. 83° 47′ W.	tang.	10.06268	sec.	10.96524
dist.	= 8732 miles.				3.94113

3. A ship sails from New York a distance of $675\frac{1}{2}$ miles, upon a course S. E. $\frac{1}{4}$ S.; find the place at which it arrives.

Ans. Three miles to the west of Georgetown in Bermuda.

4. Find the bearing and distance of Portland (Maine) from New Orleans.

Ans. The bearing = N. 49° 24′ E.
The distance = 1257 miles.

5. A ship from the Cape of Good Hope sails northwesterly until its latitude is 22° 3′ S., and its departure 3110 miles; find its course, distance sailed, longitude, and its distance from Cape St. Thomas (Brazil).

Ans. Course = N. 76° 38′ W.
Distance = 3197 miles.
Longitude = 40° 36′ W.
Distance to the Cape St. Thomas = 22 miles.

6. A ship sails from Boston upon a course E. by N. until it arrives in latitude 45° 21′ N.; find the distance, its longitude, and its distance and bearing from Liverpool.

Ans. Distance sailed = 923 miles.
Longitude = 50° 10′ W.
Distance from Liverpool = 1905 miles.
Bearing from Liverpool = S. 75° 22′ W.

7. A ship sails southwesterly from Gibraltar a distance of 1500 miles, when it is in latitude 14° 44′ N.; find its course and longitude, and distance from Cape Verde.

Ans. Course = S. 31° 17′ W.
Longitude = 19° 49′ W.
Dist. from Cape Verde = 132 miles.

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8. A ship sails from Nantucket upon a course S. 62° 11′ E., until its departure is 2274 miles; find the distance sailed, and the place arrived at.

Ans. Distance = 2571 miles.

The place arrived at is 261 miles north of Santa Cruz.

9. A ship sails southwesterly from Land's End (England) a distance of 3461 miles, when its departure is 3300 miles; find the course and the place arrived at.

Ans. The course = S. 72° 27′ W.

The place arrived at is Charleston (South Carolina).

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CHAPTER V.

MERCATOR'S SAILING.

41. THE object of *Mercator's Sailing* is to give an accurate method of calculating the difference of longitude. [B. p. 78.]

42. *Problem. To find the difference of longitude, when the distance, the course, and one latitude are known.*

Solution. Let AB (fig. 16) be the ship's track. Divide it into the small portions $A\,a$, $a\,b$, $b\,c$, &c., which are such that the difference of longitude is the same for each of them, and let

$d =$ this small difference in longitude.

Let also

$L =$ the latitude of A,

$L' =$ the latitude of B,

$l =$ the latitude of one of the points of division as b,

$l' =$ the latitude of the next point c,

$C =$ the course.

The distance $b\,c$ may then be supposed so small, that the formulas of middle latitude sailing may be applied to it; and (232) gives

$$d = (l' - l) \times \text{tang. } C \times \text{sec. } \tfrac{1}{2}\,(l' + l), \tag{239}$$

or

$$\tfrac{1}{2}\,d \text{ cotan. } C = \frac{\tfrac{1}{2}\,(l' - l)}{\text{cos. } \tfrac{1}{2}\,(l' + l)}. \tag{240}$$

But, $\frac{1}{2}(l' - l)$ is a small arc expressed in minutes, and by (14)

$$\tfrac{1}{2}(l' - l)\sin. 1' = \sin. \tfrac{1}{2}(l' - l); \tag{241}$$

which, substituted in (240), gives

$$\tfrac{1}{2} d \sin. 1' \operatorname{cotan.} C = \frac{\sin. \frac{1}{2}(l' - l)}{\cos. \frac{1}{2}(l' + l)}. \tag{242}$$

Let now

$$m = \tfrac{1}{2} d \sin. 1' \operatorname{cotan.} C = \frac{\sin. \frac{1}{2}(l' - l)}{\cos. \frac{1}{2}(l' + l)}; \tag{243}$$

and (243) may be written in the usual form of a proportion

$$\sin. \tfrac{1}{2}(l' - l) : \cos. \tfrac{1}{2}(l' + l) = m : 1; \tag{244}$$

whence, by the theory of proportions,

$$\frac{\cos. \frac{1}{2}(l' + l) + \sin. \frac{1}{2}(l' - l)}{\cos. \frac{1}{2}(l' + l) - \sin. \frac{1}{2}(l' - l)} = \frac{1 + m}{1 - m}. \tag{245}$$

But if in (47) we put

$$A = 90° - \tfrac{1}{2}(l' + l),\ B = \tfrac{1}{2}(l' - l), \tag{246}$$

we have

$$A + B = 90° - l,\ A - B = 90° - l', \tag{247}$$

and (47) becomes

$$\frac{\cos. \frac{1}{2}(l' + l) + \sin. \frac{1}{2}(l' - l)}{\cos. \frac{1}{2}(l' + l) - \sin. \frac{1}{2}(l' - l)} = \frac{\operatorname{cotan.} (45° - \frac{1}{2} l')}{\operatorname{cotan.} (45° - \frac{1}{2} l)}; \tag{248}$$

whence, if we put

$$M = \frac{1 + m}{1 - m}, \tag{249}$$

$$\frac{\operatorname{cotan.} (45° - \frac{1}{2} l')}{\operatorname{cotan.} (45° - \frac{1}{2} l)} = M. \tag{250}$$

Hence, the successive values of cotan. $(45° - \frac{1}{2} l)$ at the points A, a, b, &c., form a geometric progression; and if

$D =$ the difference of longitude of A and B,

$n =$ the number of portions of AB;

we have, by (243),

$$n = \frac{D}{d} = \frac{D \sin. 1'}{2\, m \text{ tang. } C}, \tag{251}$$

and by the theory of geometric progression

$$\text{cotan. } (45^\circ - \tfrac{1}{2} L') = \text{cotan. } (45^\circ - \tfrac{1}{2} L)\, M^n, \tag{252}$$

and by logarithms

$$\text{log. cotan. } (45^\circ - \tfrac{1}{2} L') - \text{log. cotan. } (45^\circ - \tfrac{1}{2} L) = \text{log. } M^n. \tag{253}$$

If, lastly, we put

$$e = M^{\frac{1}{2m}} \tag{254}$$

we have

$$M^n = e^{\frac{D \sin. 1'}{\text{tang. } C}} \tag{255}$$

$$\text{log. } M^n = \frac{D \sin. 1'}{\text{tang. } C} \text{ log. } e\,; \tag{256}$$

which, substituted in (253), gives by a simple reduction

$$\left[\frac{\text{cosec. } 1'}{\text{log. } e} \text{ log. cotan. } (45^\circ - \tfrac{1}{2} L') - \frac{\text{cosec. } 1'}{\text{log. } e} \text{ log. cotan. } (45^\circ - \tfrac{1}{2} L)\right] \times \text{ tang. } C = D. \tag{257}$$

Now the value of $\frac{\text{cosec. } 1'}{\text{log. } e}$ log. cotan. $(45^\circ - \frac{1}{2} L)$ has been calculated for every mile of latitude, and inserted in tables. [B. Table III.] It is called the *Meridional Parts of the Latitude*, and the method of computing it is given in the following section.

The difference between the meridional parts of the two latitudes, when the latitudes are both north or both south, is called the Meridional Difference of Latitude; but when one of the latitudes is north and the other south, the sum of the meridional parts is the meridional difference of latitude.

Hence (257) gives

$$D = \text{diff. long.} = \text{mer. diff. lat.} \times \text{tang. course.} \tag{258}$$

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43. *Corollary.* The difference of longitude is (fig. 20) the leg DE of a right triangle, of which AD is the meridional difference of latitude, and the angle A the course; and by combining this triangle with the triangle ABC of plane sailing, *all the cases of Mercator's Sailing are reduced to the solution of these two similar right triangles.*

44. *Problem. To calculate the Table of Meridional Parts.*

I. In finding the value of e, the portions of the distance are supposed to be infinitely small, hence m is by (243) also infinitely small, and its reciprocal is infinitely great.

If $1 + m$ is divided by $1 - m$, as follows,

$$\begin{array}{l} 1 - m)\ 1 + m\ (1 + 2\,m + 2\,m^2 + \&c. \\ \qquad\quad \underline{1 - m} \\ \qquad\qquad + 2\,m \\ \qquad\qquad\quad \underline{2\,m - 2\,m^2} \\ \qquad\qquad\qquad\quad + 2\,m^2 \\ \qquad\qquad\qquad\qquad \underline{2\,m^2 - 2\,m^3} \\ \qquad\qquad\qquad\qquad\qquad + 2\,m^3 \end{array}$$

we have by (249)

$$M = 1 + 2\,m + 2\,m^2 + \&c. \tag{259}$$

But since m is infinitely small, m^2, m^3, &c. are infinitely smaller, and the error of rejecting them in (259) is less than any assignable quantity; whence

$$M = 1 + 2\,m, \tag{260}$$

and by (254)

$$e = (1 + 2\,m)^{\frac{1}{2\,m}}$$

whence e has the same value as in (163 and 164).

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II. The value e (164) gives by (13)

$$\frac{\text{cosec. } 1'}{\text{log. } e} = \frac{3437.7}{\text{log. } (2.71828)} = \frac{3437.7}{0.43429} = 7915.7, \qquad (261)$$

so that we have by (257)

$$\text{Mer. parts of } L = 7915.7 \text{ log. cotan. } (45° - \tfrac{1}{2} L), \qquad (262)$$

which agrees with the explanation of Table III. given in the Preface to the Navigator.

45. Examples.

1. Calculate the meridional parts of latitude 45° 48′.

Solution. 2)45° 48′

$45° - \frac{1}{2} L = 45° - 22° 54' = 22° 6'$

22° 6′	log. cotan.	0.39141	log.	9.59263
		7915.7		3.89849
mer. parts of 45° 48′ = 3098				3.49112

2. Calculate the meridional parts of latitude 28° 14′.

Ans. 1767.

3. Calculate the meridional parts of latitude 83° 59′.

Ans. 10127.

46. *Problem. To calculate the correction for middle latitude sailing.*

Solution. If the angle DBC (fig. 19) were exactly what it should be in order that the hypothenuse BD should be the difference of longitude, and the leg BC the departure, it would be the corrected middle latitude, and we should have

$$\begin{aligned}\text{diff. long.} &= \text{sec. cor. mid. lat.} \times \text{departure} \\ &= \text{sec. cor. mid. lat.} \times \text{diff. lat.} \times \text{tang. course}, \qquad (263)\end{aligned}$$

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which, compared with (258) gives, by dividing by tang. course,

$$\text{mer. diff. lat.} = \text{sec. cor. mid. lat.} \times \text{diff. lat.} \tag{264}$$

whence
$$\text{sec. cor. mid. lat.} = \frac{\text{mer. diff. lat.}}{\text{diff. lat.}}. \tag{265}$$

If, from the corrected middle latitude, calculated by this formula, the actual middle latitude is subtracted, the correction of the middle latitude is obtained, as in the table on p. 76 of the Navigator. The meridional difference of latitude should be obtained for these calculations, not from the tables of meridional parts, but directly from the tables of logarithmic sines, &c., by means of (257); and when the difference of latitude is less than 14°, tables should be used in which the logarithms are given to seven places of decimals.

47. *Corollary.* A formula, adapted to calculation by logarithms of five places, can be obtained by the following process.

$$\text{Let } L_0 = \text{the middle latitude} = \tfrac{1}{2}(L + L')$$

$$x = \text{the correction of mid. lat.}$$

$$l_0 = \text{the difference of latitude} = L' - L,$$

and, by § 42,

$$\text{mer. diff. lat.} = \frac{\text{cosec. } 1'}{\text{log. } e} \text{ log. } \frac{\text{cotan. } (45° - \frac{1}{2} L')}{\text{cotan. } (45° - \frac{1}{2} L)}. \tag{266}$$

By changing, in (248), the small letters to large ones, we obtain

$$\text{log. } \frac{\text{cotan. } (45° - \frac{1}{2} L')}{\text{cotan. } (45° - \frac{1}{2} L)} = \text{log. } \frac{\cos. L_0 + \sin. \frac{1}{2} l_0}{\cos. L_0 - \sin. \frac{1}{2} l_0}$$

$$= \text{log. } \frac{1 + \sin. \frac{1}{2} l_0 \sec. L_0}{1 - \sin. \frac{1}{2} l_0 \sec. L_0}. \tag{267}$$

But, by (186),

$$\text{log. } \frac{1 + \sin. \frac{1}{2} l_0 \sec. L_0}{1 - \sin. \frac{1}{2} l_0 \sec. L_0} = 2 \text{ log. } e \, [\sin. \tfrac{1}{2} l_0 \sec. l_0$$

$$+ \tfrac{1}{3} (\sin. \tfrac{1}{2} l_0 \sec. L_0)^3 + \tfrac{1}{5} (\sin. \tfrac{1}{2} l_0 \sec. L_0)^5 + \&c.] \tag{268}$$

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which gives, by substitution in (267 and 266),

$$\text{mer. diff. lat.} = 2 \text{ cosec. } 1' \, [\sin. \tfrac{1}{2} l_0 \sec. L_0 + \tfrac{1}{3} (\sin. \tfrac{1}{2} l_0 \sec. L_0)^3 + \&c.] \quad (269)$$

and (265) gives

$$\sec. (L_0 + x) = \frac{\text{cosec. } 1'}{\frac{1}{2} l_0} [\sin. \tfrac{1}{2} l_0 \sec. L_0 + \tfrac{1}{3} (\sin. \tfrac{1}{2} l_0 \sec. L_0)^3 + \&c.] \quad (270)$$

48. Examples.

1. Find the correction for middle latitude sailing, when the middle latitude is 35°, and the difference of latitude 14°.

Solution. Greater lat. = 35° + 7° = 42°

Less lat. = 35° — 7° = 28°

45° — ½ gr. lat. = 24° cotan.	0.35142		
45° — ½ less lat. = 31° cotan.	0.22123		
	0.13019	log.	9.11458
	7915.7		3.89849
diff. lat. = 840′		ar. co.	7.07572
corrected mid. lat. = 35° 24′		sec.	10.08879

correction = 35° 24′ — 35° = 24′.

2. Find the correction for middle latitude sailing, when the middle latitude is 66°, and the difference of latitude 10°.

Solution. In this case $\frac{1}{2} l_0 = 5° = 300'$, $L_0 = 66°$

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	5° sin.		8.94030	
	66° sec.		0.39069	
sin. 5° sec. 66°	= 0.21428		9.33099	0.21428
(sin. 5° sec. 66°)^3	= 0.00984		7.99297	$\frac{1}{3}$ (0.00984) = 0.00328
(sin. 5° sec. 66°)^5	= 0.00045		6.65495	$\frac{1}{5}$ (0.00045) = 0.00009
(sin. 5° sec. 66°)^7	= 0.00002		5.31693	$\frac{1}{7}$ (0.00002) = 0.00000
(0.21765)	log.	9.33776		0.21765
300′	ar. co.	7.52288		
1′	cosec.	3.53627		
66° 22′	sec.	0.39691		

Corr. mid. lat. 66° 22′ — 66° = 22′.

3. Find the correction for middle latitude sailing, when the middle latitude is 30°, and the difference of latitude 4°.

Solution. In this case $\frac{1}{2}\,l_0 = 2° = 120'$, $L_0 = 30°$

	2° sin.	8.54282	
	30° sec.	.06247	
sin. 2° sec. 30°	= 0.040298	8.60529	0.040298
(sin. 2° sec. 30°)^2	= 0.000065	5.81587	0.000022
		8.60552	0.040320
	120′ ar. co.	7.92082	
	1′ cosec.	3.53627	
	30° 2′ sec.	0.06261	

Corr. mid. lat. = 30° 2′ — 30° = 2′.

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4. Find the correction for middle latitude sailing, when the middle latitude is 60°, and the difference of latitude 16°.

Ans. 46′.

5. Find the correction for middle latitude sailing, when the middle latitude is 72°, and the difference of latitude 20°.

Ans. 124′.

6. Find the correction for middle latitude sailing, when the middle latitude is 50°, and the difference of latitude 8°.

Ans. 9′.

7. Find the correction for middle latitude sailing, when the middle latitude is 68°, and the difference of latitude 12°.

Ans. 34′.

8. Find the correction for middle latitude sailing, when the middle latitude is 21°, and the difference of latitude 3°.

Ans. 1′.

9. Find the correction for middle latitude sailing, when the middle latitude is 24°, and the difference of latitude 6°.

Ans. 5′.

10. Find the correction for middle latitude sailing, when the middle latitude is 15°, and the difference of latitude 12°.

Ans. 26′.

49. *Problem. To find the bearing and distance of two given places.* [B. p. 79.]

Solution. We have by (fig. 20) for the bearing,

$$\text{tang. bearing} = \frac{\text{diff. long.}}{\text{mer. diff. lat.}}, \tag{271}$$

and the distance is found by (230).

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50. *Problem. To find the course, distance, and difference of longitude, when both latitudes and the departure are given.* [B. p. 80.]

Solution. The course is found by (229), the difference of longitude by (258), and the distance by (230).

51. *Problem. To find the distance and difference of longitude, when both latitudes and the course are given.* [B. p. 82.]

Solution. The distance is found by (230), and the difference of longitude by (258).

52. *Problem. To find the course and difference of longitude, when both latitudes and the distance are given.* [B. p. 83.]

Solution. The course is found by (233), and the difference of longitude by (258).

53. *Problem. To find the distance, the difference of latitude, and the difference of longitude, when one latitude, the course, and departure are given.* [B. p. 84.]

Solution. The distance is found by (236), the difference of latitude by (235), and the difference of longitude by (258).

54. *Problem. To find the course, the difference of latitude, and the difference of longitude, when one latitude, the distance, and the departure are given.* [B. p. 85.]

Solution. The course is found by (237), the difference of latitude by (238), and the difference of longitude by (258), or by the following proportion deduced from the similar triangles of (fig. 20).

$$\text{diff. lat.} : \text{dep.} = \text{mer. diff. lat.} : \text{diff. long.} \qquad (272)$$

55. EXAMPLES.

1. A ship sails from Boston a distance of 6747 miles, upon a course S. 46° 59½′ E.; to find the place at which it arrives.

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Solution.

Dist.	$= 6747$		3.82911	
Course	$= 46^\circ\ 59\frac{1}{2}'$	cos.	9.83385	tang. 10.03022
Diff. lat.	$= 76^\circ\ 42'$ S. $= 4602'$,		3.66296	mer. d. l. $= 5007$, 3.69958
Lat. left	$= 42^\circ\ 21'$ N.	mer. p.	2810	
Lat. in	$= 34^\circ\ 21'$ S.		2197	diff. long. $= 5368'$ 3.72980
				$= 89^\circ 28'$ E.
		mer. diff. lat. $=$	5007	long. left $= 71^\circ\ 5'$ W.
				long. in $18^\circ 23'$ E.

Ans. The place arrived at is the Cape of Good Hope.

2. Find the bearing and distance from Moscow to St. Helena.

Solution.

Moscow,	lat. $55^\circ\ 46'$ N.	mer. parts	4049	long. $37^\circ\ 33'$ E.
St. Helena,	lat. $15^\circ\ 55'$ S.	mer. parts	968	long. $5^\circ\ 36'$ W.
Diff. lat.	$= 71^\circ\ 41'$	mer. diff. lat.	5017	diff. l. $= 43^\circ\ 9'$
	$= 4301'$			$= 2589'$
Mer. diff. lat.	$= 5017$	(ar. co.)	6.29956	
Diff. long.	$= 2589$		3.41313	
Course	$=$ S. $27^\circ\ 18'$ W.	tang.	9.71269	sec. 10.05127
		diff. lat. $=$	4301	3.63357
		dist. $=$	4840 miles	3.68484

3. A ship sails from a position 200 miles to the east of Cape Horn a distance of 3635 miles, upon a course N. N. E.; find the position at which it has arrived.

Ans. It has arrived at the equator in the longitude of $33^\circ\ 18'$ W.

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4. Required the bearing and distance of Botany Bay from London.

Ans. The Bearing $=$ S. 57° 31′ E.

Distance $=$ 9551 miles.

5. A ship sailed northwesterly from Lima until it arrives in the latitude 23° 7′ N., and has made a departure of 9983 miles; find the place at which it has arrived.

Ans. Canton.

6. A ship sails from Disappointment Island in the North Pacific Ocean, upon a course S. 61° 41′ E., until it has arrived in latitude 14° 7′ S.; find the place at which it has arrived.

Ans. The Disappointment Islands in the South Pacific Ocean.

7. A ship sails from Icy Cape (North West Coast of America) a distance of 9138 miles southeasterly, when it has arrived in latitude 62° 30′ S.; find the place at which it has arrived.

Ans. Yankee Straits in New South Shetland.

8. A ship sails from Java Head, upon a course S. 68° 53′ W., until it has made a departure of 4749 miles; find the position at which it has arrived.

Ans. It has arrived at a position 180 miles south of the Cape of Good Hope.

9. A ship sails southeasterly from the South Point of the Great Bank of Newfoundland a distance of 2821 miles, when it has made a departure of 910 miles; find the position at which it has arrived.

Ans. Its position is 208 miles north of Cape St. Roque.

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CHAPTER VI.

SURVEYING.

56. THE object of *Surveying* is to determine the dimensions and areas of portions of the earth's surface. In the application of Plane Trigonometry, the portions of the earth are supposed to be so small that the curvature of the earth is neglected. They are, in this case, nothing more than common fields bounded by lines either straight or curved.

57. *Problem. To find the area of a triangular field, when its angles and one of its sides are known.*

Solution. Let ABC (fig. 2) be the triangle to be measured, and c the given side. The area of the triangle is equal to half the product of its base by its altitude, or

$$\text{area of } ABC = \tfrac{1}{2}\, b\, p. \tag{273}$$

But, by (130),

$$\sin. C : \sin. B :: c : b,$$

whence

$$b = \frac{c \sin. B}{\sin. C};$$

and, by (131),

$$p = c \sin. A.$$

Substituting in (273), we have

$$\text{area of } ABC = \frac{c^2 \sin. A \sin. B}{2 \sin. C}. \tag{274}$$

58. *Problem. To find the area of a triangular field, when two of its sides and the included angle are known.*

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Solution. Let ABC (fig. 2) be the triangle to be measured, b and c the given sides, and A the given angle. Then, by (273),

$$\text{area of } ABC = \tfrac{1}{2}\, b\, p,$$

and, by (131),

$$p = c \sin. A.$$

Hence

$$\text{area of } ABC = \tfrac{1}{2}\, b\, c \sin. A\,; \qquad (275)$$

or, *the area of a triangle is equal to half the continued product of two of its sides and the sine of the included angle.*

59. *Problem. To find the area of a triangular field, when its three sides are known.*

Solution. Let ABC (fig. 1) be the given triangle. Then, by (275),

$$\text{area of } ABC = \tfrac{1}{2}\, b\, c \sin. A\,;$$

but, by (158),

$$\sin. A = \frac{2\sqrt{[s\,(s-a)\,(s-b)\,(s-c)]}}{b\,c},$$

in which s denotes the half sum of the three sides of the triangle.

Hence

$$b\, c \sin. A = 2\sqrt{[s\,(s-a)\,(s-b)\,(s-c)]}\,;$$

and

$$\text{area of } ABC = \sqrt{[s\,(s-a)\,(s-b)\,(s-c)]}\,; \qquad (276)$$

or, *to find the area of a triangular field, subtract each side separately from the half sum of the sides; and the square root of the continued product of the half sum and the three remainders is the required area.*

60. Examples.

1. Given the three sides of a triangular field, equal to 45.56 ch., 52.98 ch., and 61.22 ch.; to find its area.

Solution. In (fig. 1) let $a = 45.56$ ch., $b = 52.98$ ch., $c = 61.22$ ch.

$2\,s =$	159.76 ch.	
$s =$	79.88 ch.	1.90244
$s - a =$	34.32 ch.	1.53555
$s - b =$	26.90 ch.	1.42975
$s - c =$	18.66 ch.	1.27091
	2	6.13865
Area of $ABC =$	1173.05 sq. ch.	3.06932.

Ans. The area = 117 A. 1 R. 9 r.

2. Given the three sides of a triangular field equal to 32.56 ch., 57.84 ch., and 44.44 ch.; to find its area.

Ans. The area = 71 A. 3 R. 29 r.

3. Given one side of a triangular field equal to 17.95 ch., and the adjacent angles equal to 100° and 70°; to find its area.

Ans. The area = 85 A. 3 R. 17 r.

4. Given two sides of a triangular field equal to 12.34 ch., and 17.97 ch., and the included angle equal to 44° 56′; to find its area.

Ans. The area = 7 A. 3 R. 13 r.

61. *Problem. To find the area of an irregular field bounded by straight lines.*

First Method of Solution. Divide the field into triangles in any manner best suited to the nature of the ground. Measure all those sides and angles which can be measured conveniently, remembering that three parts of each triangle, one of which is a side, must be known to determine it.

But it is desirable to measure more than three parts of each triangle, when it can be done; because the comparison of them with each other will often serve to correct the errors of observation. Thus, if the three angles were measured, and their sum found to

differ from 180°, it would show there was an error; and the error, if small, might be divided between the angles; but if the error was large, it would show the observations were inaccurate, and must be taken again.

The area of each triangle is to be calculated by one of the preceding formulas, and the sum of the areas of the triangles is the area of the whole field.

This method of solution is general, and may be applied to surfaces of any extent, provided each triangle is so small as not to be affected by the earth's curvature.

Second Method of Solution. Let $ABCEFH$ (fig. 21) be the field to be measured. Starting from its most easterly or its most westerly point, the point A for instance, measure successively round the field the bearings and lengths of all its sides. Through A draw the meridian NS, on which let fall the perpendiculars BB', CC', EE', FF', and HH'. Also draw $CB''E''$, EF'', and HF''' parallel to NS.

Then the area of the required field is

$$ABCEFH = AC'CEFF' - [AC'CB + AHFF'].$$

But

$$AC'CEFF' = C'CEE' + E'EFF';$$

and

$$AC'CB + AHFF' = C'CBB' + B'BA + AHH' + H'HFF'.$$

Hence

$$ABCEFH = [C'CEE' + E'EFF'] - [C'CBB' + B'BA + AHH' + H'HFF'];$$

or doubling and changing a very little the order of the terms,

$$\left.\begin{array}{l} 2\,ABCEFH = [2\,C'CEE' + 2\,E'EFF'] - \\ [2\,B'BA + 2\,CC'BB' + 2\,H'HFF' + 2\,AHH']. \end{array}\right\} \quad (277)$$

Again,

$$\left.\begin{array}{lll} 2\,B'BA & = BB' & \times AB' \\ 2\,CC'BB' & = (BB' + CC') & \times B'C' \\ 2\,C'CEE' & = (EE' + CC') & \times E'C' \\ 2\,E'EFF' & = (EE' + FF') & \times E'F' \\ 2\,H'HFF' & = (HH' + FF') & \times H'F' \\ 2\,AHH' & = HH' & \times AH'. \end{array}\right\} \quad (278)$$

So the determination of the required area is now reduced to the calculation of the several lines in the second members of (278). But the rest of the solution may be more easily comprehended by means of the following table, which is precisely similar in its arrangement to the table actually used by surveyors, when calculating areas by this process.

Sides.	N.	S.	E.	W.	Dep.	Sum.	N. Areas.	S. Areas.
AB	AB′		BB′		BB′	BB′	BB′A	
BC	B′C′			BB″	CC′	BB′ + CC′	CC′BB′	
CE		C′E′	EE″		EE′	CC′ + EE′		C′CEE′
EF		E′F′	FF″		FF′	EE′ + FF′		E′EFF′
FH	F′H′			FF‴	HH′	FF′ + HH′	H′HFF′	
HA	H′A′			HH′	O	HH′	AHH′	

In the *first* column of the table are the successive sides of the field.

In the *second* and *third* columns are the differences of latitude of the several sides, the column headed N, corresponding to the sides running in a northerly direction, and that headed S, corresponding to those running in a southerly direction.

These two columns are calculated by the formula

$$\text{Diff. lat.} = \text{dist.} \times \text{cos. bearing.}$$

In the *fourth* and *fifth* columns are the departures of the several sides; the column headed E, corresponding to the sides running in an easterly direction, and that headed W, to those running in a westerly direction.

These two columns are calculated by the formula

$$\text{Departure} = \text{dist.} \times \text{sin. bearing.}$$

In the *sixth* column, headed *Departure*, are the departures of the several vertices, which terminate each side of the field from the vertex *A*. This column is calculated from the two

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columns E and W, in the following manner. *The first number in column Departure is the same as the first in the two columns E and W; and every other number in column Departure is obtained by adding the corresponding number in columns E and W, if it is of the same column with the first number in those two columns, to the previous number in column Departure; and by subtracting it, if it is of a different column.*

Thus,

$$BB' = BB'$$

$$CC' = B'B'' = BB' - BB''$$

$$EE' = E'E'' + EE'' = CC' + EE''$$

$$FF' = F'F' + FF'' = EE' + FF''$$

$$HH' = F'F''' = FF' - FF'''$$

$$O = HH' - HH'.$$

In the *seventh* column, headed *Sum*, are the first factors of the second members of (278). This column is calculated from column Departure in the following manner. *The first number in column Sum is the same as the first in column Departure; every other number in column Sum is the sum of the corresponding number in column Departure added to the previous number in column Departure*, as is evident from simple inspection.

In the *eighth* and *ninth* columns are the values of the areas, which compose the first members of (278). *These columns are calculated by multiplying the numbers in column Sum by the corresponding numbers in columns N and S*, which contain the second factors of the second members of (278). *The products are written in the column of North Areas, when the second factors are taken from column N, and in that of South Areas, when the second factors are taken from column S.*

If we compare the columns of North and South Areas with (277), we find that all those areas, which are preceded by

the negative sign, are the same with those in the column of North Areas; while all those, which are connected with the positive sign, belong to the column of South Areas. *To obtain, therefore, the value* of the second member of (277), that is, *of double the required area, we have only to find the difference between the sums of the columns of North and South Areas.* [B. p. 107.]

62. *Corollary.* The columns N, S, E, and W, are those which would be calculated in Traverse Sailing, if a ship was supposed to start from the point A, and proceed round the sides of the field till it returned to the point A. The difference of the sums of columns N and S is, then, by traverse sailing, the difference of latitude between the point from which the ship starts, and the point at which it arrives; and the difference of columns E and W is the departure of the same two points. But as both the points are here the same, their difference of latitude and their departure must be nothing, or

$$\text{Sum of column N} = \text{sum of column S};$$

$$\text{Sum of column E} = \text{sum of column W}.$$

But when, as is almost always the case, the sums of these columns differ from each other, the difference must arise from errors of observation. If the error is great, new observations must be taken; but if it is small, it may be divided among the sides by the following proportion.

$$\textit{The sum of the sides} : \textit{each side} = \textit{whole error} :$$

$$\textit{error corresponding to each side.} \qquad (279)$$

The errors corresponding to the sides are then to be subtracted from the differences of latitude, or the departures which are in the larger column, and added to those which are in the smaller column.

63. Examples.

1. Given the bearings and lengths of the sides of a field, as in the three first columns of the following table, to find its area.

Solution. The table is computed by § 61.

No.	Bearing.	Dist.	N.	S.	E.	W.	Cor. N.	Cor. E.	N.	S.	E.	W.	Dep.	Sum.	N. Areas.	S. Areas.
1	N.45° W.	21 ch.	14.85			14.85	.02	.05	14.83			14.90	14.90	14.90	220.9670	
2	N.24° E.	32 ch.	29.24		13.01		.03	.08	29.21		12.93		1.97	16.87	492.7727	
3	S.86° W.	54 ch.		3.76		53.87	.05	.14		3.81		54.01	55.98	57.95		220.7895
4	South.	10 ch.		10.00			.01	.03		10.01		.03	56.01	111.99		1121.0199
5	S.70° W.	11 ch.		3.76		10.34	.01	.03		3.77		10.37	66.38	122.39		461.4103
6	S.20° E.	99 ch.		93.03	33.86		.10	.26		93.13	33.60		32.78	99.16		9234.7708
7	East.	6 ch.			6.00		.00	.02			5.98		26.80	59.58		
8	N.22° E.	72 ch.	66.76		26.98		.08	.18	66.68		26.80		0.00	26.80	1787.0240	
			110.85	110.55	79.85	79.06	.30	.79	110.72	110.72	79.31	79.31			2500.7637	11037.9905
			110.55		79.06											2500.7637
			.30 N.		.79 E.											2)8537.2268
																10)4268.6134
																426.8613
																4
																3.4452
																40
																17.8080

Ans. The Area = 426 A. 3 R. 18 r.

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2. Given the lengths and bearings of the sides of a field, as in the following table; to find its area.

No.	Bearings	Dist.
1	N. 17° E.	25 ch.
2	East.	28 ch.
3	South.	54 ch.
4	S. 4° W.	22 ch.
5	N. 33° W.	62 ch.

Ans. The area = 169 A. 3 R. 17 r.

64. *Problem. To find the area of a field bounded by sides, irregularly curved.*

Solution. Let $ABCEFHIKL$ (fig. 22) be the field to be measured, the boundary $ABCEFHIKL$ being irregularly curved. Take any points C and F, so that by joining AC, CF, and FL, the field $ACFL$, bounded by straight lines, may not differ much from the given field.

Find the area of $ACFL$, by either of the preceding methods, and then measure the parts included between the curved and the straight sides by the following method of *offsets.*

Take the points a, b, c, d, so that the lines $A\,a$, $a\,b$, $b\,c$, $c\,d$, $d\,C$ may be sensibly straight. Let fall on AC the perpendiculars $a\,a'$, $b\,b'$, $c\,c'$, $d\,d'$. Measure these perpendiculars, and also the distances $A\,a'$, $a'\,b'$, $b'\,c'$, $c'\,d'$, $d'\,C$.

The triangles $A\,a\,a'$, $C\,d\,d'$, and the trapezoids $a\,b\,a'\,b'$, $b\,c\,b'\,c'$, $c\,d\,c'\,d'$ are then easily calculated, and their sum is the area of ABC.

In the same way may the areas of CEF, FHI, and IKL be calculated; and then the required area is found by the equation

$$ABCEFHIKL = ACFL - ABC + CEF + FHI - IKL.$$

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EXAMPLE.

Given (fig. 22) $A\,a' = 5$ ch., $a'\,b' = 2$ ch., $b'\,c' = 6$ ch., $c'\,d' = 1$ ch., $d'\,C = 4$ ch.; also $a\,a' = 3$ ch., $b\,b' = 2$ ch., $c\,c' = 2.5$ ch., $d\,d' = 1$ ch.; to find the area of ABC.

Ans. Required area = 2 A. 3 R. 36 r.

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CHAPTER VII.

HEIGHTS AND DISTANCES.

65. THE plane of the *sensible horizon* at any place, is the tangent plane to the earth's surface at that place. [B. p. 48.]

The horizontal plane coincides with that of the surface of tranquil waters, when this surface is so small that its curvature may be neglected; and it is perpendicular to the *plumb line.*

66. The *angle of elevation* of an object is the angle which the line drawn to it makes with the horizontal plane, when the object is above the horizon; the *angle of depression* is the same angle when the object is below the horizon.

67. The *bearing of one object from another* is the angle included by the two lines which are drawn from the observer to these two objects.

68. *Problem. To determine the height of a vertical tower, situated on a horizontal plane.* [B. p. 94.]

Solution. Observation. Let AB (fig. 23) be the tower, whose height is to be determined. Measure off the distance BC on the horizontal plane of any convenient length. At the point C observe the angle of elevation ACB.

Calculation. We have, then, given in the right triangle ACB the angle C and the base BC, as in problem § 34 of Pl. Trig., and the leg AB is found by (26).

EXAMPLE.

At the distance of 95 feet from a tower, the angle of elevation of the tower is found to be 48° 19′. Required the height of the tower.

Ans. 106.69 feet.

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69. *Problem. To find the height of a vertical tower situated on an inclined plane.*

Solution. Observation. Let AB (fig. 24) be the tower situated on the inclined plane BC. Observe the angle B, which the tower makes with the plane. Measure off the distance BC of any convenient length. Observe the angle C, made by a line drawn to the top of the tower with BC.

Calculation. In the oblique triangle ABC, there are given the side BC and the two adjacent angles B and C, as in § 73 of Pl. Trig.

Example.

Given (fig. 24) $BC = 89$ feet, $B = 113° \, 12'$, $C = 23° \, 27'$; to find AB.

Ans. $AB = 51.595$ feet.

70. *Problem. To find the distance of an inaccessible object.* [B. p. 89 and 95.]

Solution. Observation. Let B (fig. 2) be the point, the distance of which is to be determined, and A the place of the observer. Measure off the distance AC of any convenient length, and observe the angles A and C.

Calculation. AB and BC are found by § 73 of Pl. Trig.

71. *Corollary.* The perpendicular distance BP of the point B from the line AC, and the distances AP and PC are found in the triangles ABP and BPC, by § 32 of Pl. Trig.

72. *Corollary.* Instead of directly observing the angles A and C, the bearings of the lines AB, AC, and BC, may be observed, when the plane ABC is horizontal, and the angles A and C are easily determined.

73. Examples.

1. An observer sees a cape, which bears N. by E.; after sailing

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30 miles N. W., he sees the same cape bearing east; find the distance of the cape from the two points of observation.

Ans. The first distance $= 21.63$ miles.

The second dist. $= 25.43$ miles.

2. Two observers, stationed on opposite sides of a cloud, observe the angles of elevation to be $44^\circ\ 56'$, and $36^\circ\ 4'$, their distance apart being 700 feet; find the distance of the cloud from each observer, and its perpendicular altitude.

Ans. Distances from observers $= 417.2$ feet, and $= 500.6$ ft.

Height $= 294.7$ feet.

3. The angle of elevation of the top of a tower at one station is observed to be $68^\circ\ 19'$, and at another station, 546 feet farther from the tower, the angle of elevation is $32^\circ\ 34'$; find the height and distance of the tower, the two points of observation being supposed to be in the same horizontal plane with the foot of the tower.

Ans. The height . . . . . . $= 467.45$ ft.

The distance from the nearest point of observ. $= 185.86$ ft.

74. *Problem. To find the distance of an object from the foot of a tower of known height, the observer being at the top of the tower.*

Solution. Observation. Let the tower be AB (fig. 23), and the object C. Measure the angle of depression HAC.

Calculation. Since

$$ACB = HAC,$$

we know in the triangle ACB the leg AB and the opposite angle C, as in § 33 of Pl. Trig.

EXAMPLE.

Given the height of the tower $= 150$ feet, and the angle of depression $= 17^\circ\ 25'$; to find the distance from the foot of the tower.

Ans. 478.16 feet.

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75. *Problem. To find the height of an inaccessible object above a horizontal plane, and its distance from the observer.* [B. p. 96.]

Solution. Observation. Let A (fig. 25) be the object. At two different stations, B and C, whose distance apart and bearing from each other are known, observe the bearings of the object, and also the angle of elevation at one of the stations, as B.

Calculation. In the triangle BCD, the side BC and its adjacent angles are known, so that BD is found by § 73 of Pl. Trig. In the right triangle ABD, the height AD is, then, computed by § 34 of Pl. Trig.

EXAMPLE.

At one station the bearing of a cloud is N. N. W., and its angle of elevation 50° 35′. At a second station, whose bearing from the first station is N. by E., and distance 5000 feet, the bearing of the cloud is W. by N.; find the height of the cloud.

Ans. 7316.3 feet.

76. *Problem. To find the distance of two objects, whose relative position is known.* [B. p. 90.]

Solution. Observation. Let B and C (fig. 1) be the two known objects, and A the position of the observer. Observe the bearings of B and C from A.

Calculation. In the triangle ABC, the side BC and the two angles are known. The sides of AB and AC are found by § 73 of Pl. Trig.

EXAMPLE.

The bearings of the two objects are, of the first N. E. by E., and of the second E. by S.; the known distance of the first object from the second is 23.25 miles, and the bearing N. W.; find their distance from the observer.

Ans. The distance of the first object is = 18.27 miles.

That of the second object = 32.25 miles.

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77. *Problem. To find the distance apart of two objects separated by an impassable barrier.* [B. p. 91.]

Solution. Observation. Let A and B (fig. 1) be the objects; the distance of which from each other is sought. Measure the distances and bearings from any point C, to both A and B.

Calculation. In the triangle ABC, the two sides AC and BC, and the included angle C, are known. The side AB and the angles A and B may be found by § 82 of Pl. Trig.

EXAMPLE.

Two ships sail from the same port, the one N. 10° E. a distance of 200 miles, the second N. 70° E. a distance of 150 miles; find their bearing and distance.

Ans. The distance . . . . . = 180.3 miles.

The bearing of the first ship from the second = N. 36° 6′ W.

78. *Problem. To find the distance apart of two inaccessible objects situated in the same plane with the observer, and their bearing from each other.* [B. p. 92.]

Solution. Observation. Let A and B (fig. 26) be the two inaccessible objects. At two stations, C and E, observe the bearings of A and B, and the bearing and distance of C from E.

Calculation. In the triangle AEC, we have the side CE, and the angles ACE and AEC, so that AC is found by § 73 of Pl. Trig.

In the same way BC is calculated from the triangle BCE.

Lastly, in triangle ABC, we know the two sides AC and BC, and the included angle, for

$$ACB = ACE - BCE.$$

Hence AB and the angles BAC and CBA are found by § 82.

EXAMPLE.

An observer from a ship saw two headlands; the first bore E.N.E., and the second N.W. by N. After he had sailed N. by W. 16.25

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miles, the first headland bore E. and the second N. W. by W.; find the bearing and distance of the first headland from the second.

Ans. Distance = 55.89 miles.

Bearing = S. 80° 42′ E.

79. *Problem. To find the distance of an object of known height, which is just seen in the horizon.*

Solution. I. *If light moved in a straight line*, and if A (fig. 27) were the eye of the observer, and B the object, the straight line APB would be that of the visual ray. The point P, at which the ray touches the curved surface CPD of the earth, is the point of the visible horizon at which the object is seen. The distances PA and PB may be calculated separately, when the heights AC and BD are known. For this purpose, let O be the earth's centre, let BD be produced to E, and let

$$h = AC, \ H = BD,$$

$$l = PA, \ L = PB,$$

$$R = \text{the earth's radius.}$$

Since BP is a tangent, and BOE a secant to the earth, we have

$$BE : BP = BP : BD;$$

and BD is so small in comparison with the radius, that we may take

$$BE = DE = 2\,R,$$

and the above proportion becomes

$$2\,R : L = L : H;$$

whence

$$L^2 = 2\,RH, \ L = \sqrt{(2\,RH}, \qquad (280)$$

$$H = \frac{L^2}{2\,R}; \qquad (281)$$

and in the same way

$$l^2 = 2\,R\,h, \ l = \sqrt{(2\,R\,h)}, \qquad (282)$$

$$h = \frac{l^2}{2\,R}. \qquad (283)$$

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II. Light does not, however, move in a straight line near the earth's surface, but *in a line curved towards the earth's centre, which line is nearly an arc of a circle, whose radius is seven times the earth's radius;* so that for the point of contact P and the distances l and L, the positions of the eye and of the object are A' and B'. Now if we put

$$BB' = H',\ B'D = H_1 = H - H'$$
$$A'C = h_1,$$

we can find the value of H' with sufficient accuracy by changing in (281) R into $7\,R$, which gives

$$H' = \frac{L^2}{14\,R} = \tfrac{1}{7}\,H$$

$$H_1 = H - H' = \tfrac{6}{7}\,H = \frac{3\,L^2}{7\,R} \qquad (284)$$

whence $$L = \sqrt{(\tfrac{7}{3}\,RH_1)}. \qquad (285)$$

III. In calculating the value of L by (285), it is usually desired in statute miles, while the height H_1 is given in feet. Now we have in the Preface to the Navigator, page v,

$$R = 20911790 \text{ feet}, \qquad (286)$$

whence $$\tfrac{7}{3}\,R = 48794177 \text{ feet},$$

$$\text{log.}\ \sqrt{(\tfrac{7}{3}\,R)} = \tfrac{1}{2}\ \text{log.}\ \tfrac{7}{3}\,R = 3.84418,$$

and $$\text{log.}\ (L \text{ in feet}) = 3.84418 + \tfrac{1}{2}\ \text{log.}\ (H_1 \text{ in feet}).$$

But $$L \text{ in miles} = \frac{L \text{ in feet}}{5280},$$

so that $$\text{log.}\ L \text{ in miles} = \text{log.}\ L \text{ in feet} - 3.72263$$
$$= 0.12155 + \tfrac{1}{2}\ \text{log.}\ H_1 \text{ in feet}, \qquad (287)$$

which agrees with the formula given in the Preface to the Navigator for calculating Table X.

IV. The Table may be used for finding L and l, when H_1 and h_1 are given, and then the required distance is the sum of L and l.

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80. *Corollary.* Table X gives the correction for the error which is committed in § 68, by neglecting the earth's curvature, for it is evident that to the height BP (fig. 28) of the object above the visible level must be added the height PC of the level above the curved surface of the earth, as in B. p. 95.

81. Examples.

1. Calculate the distance in Table X at which an object can be seen from the suface of the earth, when its height is 5000 feet.

Solution.

$\frac{1}{2}$ log. 5000	$= \frac{1}{2}$ (3.69897) $=$	1.84948
constant log.	$=$	0.12155
dist. $=$ 93.5 (as in Table X)		1.97103

2. Being on a hill 200 feet above the sea, I see just appearing in the horizon the top of a mast, which I know to be 150 feet above water; how far distant is it?

Solution. By Table X,

200 feet corresponds to 18.71 miles.

150 feet corresponds to 16.20 miles.

The distance is 34.19 miles.

3. At the distance of $7\frac{1}{2}$ statute miles from a hill the angle of elevation of its top is $2° 13'$; find its height in feet, the observer being 20 feet above the sea.

Solution.

	2° 13	tang. 8.58779	
$7\frac{1}{2}$ miles $=$	39600	4.59770	By Table X.
	1533 feet	3.18549	7.50
	1 foot correction,	height 20	5.92
height $=$	1534 feet,	height 1	1.58

4. Calculate the distance in Table X, when the height is 450 feet.

Ans. 28.06 miles.

5. Upon a height of 5000 feet, the top of a hill, one mile high, is just visible in the horizon; how far distant is the hill?

Ans. 189.6 miles.

6. At the distance of 25 miles from a mountain the angle of elevation of its top is 3°; find its height, the observer being 60 feet above the intervening sea.

Ans. 7043 feet.

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SPHERICAL TRIGONOMETRY.

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SPHERICAL TRIGONOMETRY.

CHAPTER I.

DEFINITIONS.

1. *Spherical Trigonometry* treats of the solution of *spherical triangles.*

A *Spherical Triangle* is a portion of the surface of a sphere included between three arcs of great circles.

In the present treatise those spherical triangles only are treated of, in which the sides and angles are less than 180°.

2. *The angle*, formed by two sides of a spherical triangle, is the same as the angle formed by their planes.

3. Besides the usual method of denoting sides and angles by degrees, minutes, &c.; another method of denoting them is so often used in Spherical Astronomy, that it will be found convenient to explain it here.

The circumference is supposed to be divided into 24 equal arcs, called *hours;* each hour is divided into 60 *minutes of time*, each minute into 60 *seconds of time*, and so on.

Hours, minutes, seconds, &c. of time are denoted by h, m, s, &c.

4. *Problem. To convert degrees, minutes, &c. into hours, minutes, &c. of time.*

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Solution. Since

$$360^\circ = 24^h$$

we have $15^\circ = 1^h$, and $1^\circ = \frac{1}{15}^h = 4^m$,

and $15' = 1^m$, and $1' = 4^s$,

$15'' = 1^s$, and $1'' = 4^t$.

Hence $a^\circ = 4\,a^m$, $a'\ 4\,a^s$, $a'' = 4\,a^t$;

so that *to convert degrees, minutes, &c. into time, multiply by* 4, *and change the marks* $^\circ\,'\,''$ *respectively, into* $^{m\,s\,t}$.

5. *Corollary. To convert time into degrees, minutes, &c., multiply the hours by* 15 *for degrees, and divide the minutes, seconds, &c. of time by* 4, *changing the marks* $^{m\,s\,t}$ *into* $^\circ\,'\,''$.

The turning of degrees, minutes, &c, into time, and the reverse, may be at once performed by table XXI of the Navigator.

6. Examples.

1. Convert $225^\circ\ 47'\ 38''$ into time.

Solution. By § 4.

$$\begin{array}{rcl} 225^\circ & = 900^m = & 15^h \\ 47' & = 188^s = & 3^m\ 8^s \\ 38'' & = 152^t = & 2^s\ 32^t \\ \hline 225^\circ\ 47'\ 38'' & = & 15^h\ 3^m\ 10^s\ 32^t \end{array}$$

By Table XXI.

$$\begin{array}{r} 15^h \\ 3^m\ 8^s \\ 2^s\ 32^t \\ \hline 15^h\ 3^m\ 10^s\ 32^t \end{array}$$

2. Convert $17^h\ 19^m\ 13^s$ into degrees, minutes, &c.

Solution. By § 5.

$$\begin{array}{rcl} 17^h & = & 255^\circ \\ 19^m\ 13^s & = & 4^\circ\ 48'\ 15'' \\ \hline 17^h\ 19^m\ 13^s & = & 259^\circ\ 48'\ 15'' \end{array}$$

By Table XXI.

$$\begin{array}{rcl} 17^h\ 16^m & = & 259^\circ \\ 3^m\ 12^s & = & 48' \\ 1^s & = & 15'' \\ \hline 17^h\ 19^m\ 13^s & = & 259^\circ\ 48'\ 15'' \end{array}$$

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3. Convert 12° 34′ 56″ into time. *Ans.* $50^m\ 19^s\ 44^t$.

4. Convert 99° 59′ 59″ into time. *Ans.* $6^h\ 39^m\ 59^s\ 56^t$.

5. Convert $3^h\ 2^m\ 12^s$ into degrees, minutes, &c.

Ans. 45° 33′.

6. Convert $11^h\ 59^m\ 59^s$ into degrees, minutes, &c.

Ans. 179° 59′ 45″.

7. When an arc is given in time, its log., sine, &c. can be found directly from Table XXVII, by means of the column headed *Hour*, *P. M.*, in which twice the time is given, so that the double of the angle must be found in this column.

The use of the table of proportional parts for these columns is explained upon page 35 of the Navigator. When the time exceeds 6^h, the difference between it and 12^h or 24^h must be used.

Examples.

1. Find the log. cosine of $19^h\ 33^m\ 11^s$.

Solution.

$$24^h - 19^h\ 33^m\ 11^s = 4^h\ 26^m\ 49^s$$

$$2 \times (4^h\ 26^m\ 49^s) = 8^h\ 53^m\ 38^s$$

$8^h\ 53^m\ 36^s$ P. M.	cos.	9.59720
prop. parts of 2^s		7
$8^h\ 53^m\ 38^s$ P. M.	cos.	9.59713

2. Find the angle in time of which the log. tang. is 10.12049.

$7^h\ 2^m\ 40^s$ P. M.	tang.	10.12026
7^s prop. parts		23
2) $7^h\ 2^m\ 47^s$ P. M.		10.12049

Ans. $3^h\ 31^m\ 23\frac{1}{2}^s$

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3. Find the log. sine of $3^h\ 12^m\ 2^s$. *Ans.* 9.87113.

4. Find the log. cosine of $11^h\ 3^m\ 13$. *Ans.* 9.98653.

5. Find the log. tang. of $15^h\ 0^m\ 9^s$. *Ans.* 10.00057.

6. Find the log. cotan. of $22^h\ 59^m\ 59^s$. *Ans.* 10.57183.

7. Find the angle in time whose log. secant is 10.23456.

Ans. $3^h\ 37^m\ 26^s$.

8. Find the angle in time whose log. cosecant is 10.12346.

Ans. $3^h\ 15^m\ 15^s$.

8. An *isosceles* spherical triangle is one, which has two of its sides equal.

An *equilateral* spherical triangle is one, which has all its sides equal.

9. A spherical *right* triangle is one which has a right angle; all other spherical triangles are called *oblique*.

We shall in spherical trigonometry, as we did in plane trigonometry, attend first to the solution of right triangles.

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CHAPTER II.

SPHERICAL RIGHT TRIANGLES.

10. *Problem. To investigate some relations between the sides and angles of a spherical right triangle.*

Solution. The importance of this problem is obvious; for, unless some relations were known between the sides and the angles, they could not be determined from each other, and there could be no such thing as the solution of a spherical triangle.

Let, then, ABC (fig. 29) be a spherical right triangle, right-angled at C. Call the hypothenuse AB, h; and call the legs BC and AC, opposite the angles A and B, respectively a and b.

Let O be the centre of the sphere. Join OA, OB, OC.

The angle A is, by art. 2, equal to the angle of the planes BOA and COA. The angle B is equal to the angle of the planes BOC and BOA. The angle of the planes BOC and AOC is equal to the angle C, that is, to a right angle; these two planes are, therefore, perpendicular to each other.

Moreover, the angle BOA, measured by BA, is equal to BA or h; BOC is equal to its measure BC or a, and AOC is equal to its measure AC or b.

Through any point A' of the line OA, suppose a plane to pass perpendicular to OA. Its intersections $A'C'$ and $A'B'$ with the planes COA and BOA must be perpendicular to OA', because they are drawn through the foot of this perpendicular.

As the plane $B'A'C'$ is perpendicular to OA, it must be perpendicular to AOC; and its intersection $B'C'$ with the plane BOC, which is also perpendicular to AOC, must likewise be perpendicular to AOC. Hence $B'C'$ must be perpendicular to $A'C'$ and OC', which pass through its foot in the plane AOC.

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All the triangles $A'OB'$, $A'OC'$, $B'OC'$, and $A'B'C'$, are then right-angled; and the comparison of the mleads to the desired equations, as follows:

First. We have from triangle $A'OB'$ by (4)

$$\cos. A'OB' = \cos. h = \frac{OA'}{OB'};$$

and from triangles $A'OC'$ and $B'OC'$

$$\cos. A'OC' = \cos. b = \frac{OA'}{OC'},$$

$$\cos. B'OC' = \cos. a = \frac{OC'}{OB'}.$$

The product of the last two equations is

$$\cos. a \cos. b = \frac{OA'}{OC'} \times \frac{OC'}{OB'} = \frac{OA'}{OB'};$$

hence, from the equality of the second members of these equations,

$$\cos. h = \cos. a \cos. b. \tag{288}$$

Secondly. From triangle $A'BC'$ we have by (4), and the fact that the angle $B'A'C'$ is equal to the inclination of the two planes AOC and BOA,

$$\cos. B'A'C' = \cos. A = \frac{A'C'}{A'B'};$$

and, from triangles $A'OC'$ and $A'OB'$, by (4),

$$\text{tang. } C'OA' = \text{tang. } b = \frac{A'C'}{A'O},$$

$$\text{cotan.} B'OA' = \text{cotan.} h = \frac{A'O}{A'B'}.$$

The product of these equations is

$$\text{tang. } b \text{ cotan. } h = \frac{A'C'}{A'O} \times \frac{A'O}{A'B'} = \frac{A'C'}{A'B'};$$

hence

$$\cos. A = \text{tang. } b \text{ cotan. } h. \tag{289}$$

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Thirdly. Corresponding to the preceding equation between the hypothenuse h, the angle A, and the adjacent side b, there must be a precisely similar equation between the hypothenuse h, the angle B, and the adjacent side a; which is

$$\cos. B = \text{tang.}\, a \,\text{cotan.}\, h. \tag{290}$$

Fourthly. From triangles $B'OC'$, $B'OA'$, and $B'A'C'$, by (4),

$$\sin. B'OC' = \sin. a = \frac{B'C'}{OB'},$$

$$\sin. B'OA' = \sin. h = \frac{B'A'}{OB'},$$

$$\sin. B'A'C' = \sin. A = \frac{B'C'}{B'A'}.$$

The product of these last two equations is

$$\sin. h \sin. A = \frac{B'A'}{OB'} \times \frac{B'C'}{B'A'} = \frac{B'C'}{OB'};$$

hence
$$\sin. a = \sin. h \sin. A. \tag{291}$$

Fifthly. The preceding equation between h, the angle A, and the opposite side a, leads to the following corresponding one between h, the angle B, and the opposite side b;

$$\sin. b = \sin. h \sin. B. \tag{292}$$

Sixthly. From triangles $C'OA'$, $B'A'C'$, and $B'OC'$, by (4),

$$\sin. C'OA' = \sin. b = \frac{A'C'}{OC'},$$

$$\text{cotan.}\, B'A'C' = \text{cotan.}\, A = \frac{A'C'}{B'C'},$$

$$\text{tang.}\, B'OC' = \text{tang.}\, a = \frac{B'C'}{OC'}.$$

The product of these last two equations is

$$\text{cotan.}\, A \,\text{tang.}\, a = \frac{A'C'}{B'C'} \times \frac{B'C'}{OC'} = \frac{A'C'}{OC'};$$

hence
$$\sin. b = \text{cotan.}\, A \,\text{tang.}\, a. \tag{293}$$

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Seventhly. The preceding equation between the angle A, the opposite side a, and the adjacent side b, leads to the following corresponding one between the angle B, the opposite side b, and the adjacent side a;

$$\sin. a = \text{cotan.}\, B \,\text{tang.}\, b. \tag{294}$$

Eighthly. From (7)

$$\text{tang.}\, a = \frac{\sin. a}{\cos. a},$$

$$\text{tang.}\, b = \frac{\sin. b}{\cos. b};$$

which, substituted in (293) and (294), give

$$\sin. a = \frac{\text{cotan.}\, B \sin. b}{\cos. b},$$

$$\sin. b = \frac{\text{cotan.}\, A \sin. a}{\cos. a}.$$

Multiplying the first of these equations by $\cos. b$, and the second by $\cos. a$, we have

$$\sin. a \cos. b = \text{cotan.}\, B \sin. b,$$

$$\sin. b \cos. a = \text{cotan.}\, A \sin. a.$$

The product of these equations is

$$\sin. a \sin. b \cos. a \cos. b = \text{cotan.}\, A \,\text{cotan.}\, B \sin. a \sin. b;$$

which, divided by $\sin. a \sin. b$, becomes

$$\cos. a \cos. b = \text{cotan.}\, A \,\text{cotan.}\, B.$$

But, by (288),

$$\cos. h = \cos. a \cos. b;$$

hence

$$\cos. h = \text{cotan.}\, A \,\text{cotan.}\, B. \tag{295}$$

Ninthly. We have, by (288) and (292),

$$\cos. a = \frac{\cos. h}{\cos. b},$$

$$\sin. B = \frac{\sin. b}{\sin. h},$$

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the product of which is, by (7) and (8),

$$\cos. a \sin. B = \frac{\sin. b \cos. h}{\cos. b \sin. h} = \frac{\sin. b}{\cos. b} \cdot \frac{\cos. h}{\sin. h}$$

$$= \text{tang. } b \text{ cotan. } h.$$

But, by (289),

$$\cos. A = \text{tang. } b \text{ cotan. } h;$$

hence $$\cos. A = \cos. a \sin. B. \tag{296}$$

Tenthly. The preceding equation between the side a, the opposite angle A, and the adjacent angle B, leads to the following similar one between the side b, the opposite angle B, and the adjacent angle A;

$$\cos. B = \cos. b \sin. A. \tag{297}$$

11. *Corollary.* The ten equations, [288–297,] have, by a most happy artifice, been reduced to two very simple theorems, called, from their celebrated inventor, *Napier's Rules.*

In these rules, the complements of the hypothenuse and the angles are used instead of the hypothenuse and the angles themselves, and the right angle is neglected.

Of the five parts, then, the legs, the complement of the hypothenuse, and the complements of the angles; either part may be called the *middle part.* The two parts, including the middle part on each side, are called the *adjacent parts;* and the other two parts are called the *opposite parts.* The two theorems are as follows.

I. *The sine of the middle part is equal to the product of the tangents of the two adjacent parts.*

II. *The sine of the middle part is equal to the product of the cosines of the two opposite parts.* [B. p. 436.]

Proof. To demonstrate the preceding rules, it is only necessary to compare all the equations which can be deduced from them, with those previously obtained. [288–297.]

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Let there be the spherical right triangle ABC (fig. 30) right-angled at C.

First. If co. h were made the middle part, then, by the above rule, co. A and co. B would be adjacent parts, and a and b opposite parts; and we should have

$$\sin. (\text{co. } h) = \text{tang. } (\text{co. } A) \text{ tang. } (\text{co. } B)$$

$$\sin. (\text{co. } h) = \cos. a \cos. b\,;$$

or

$$\cos. h = \text{cotan. } A \text{ cotan. } B,$$

$$\cos. h = \cos. a \cos. b\,;$$

which are the same as (295) and (288).

Secondly. If co. A were made the middle part; then co. h and b would be adjacent parts, and co. B and a opposite parts; and we should have

$$\sin. (\text{co. } A) = \text{tang. } (\text{co. } h) \text{ tang. } b,$$

$$\sin. (\text{co. } A) = \cos. (\text{co. } B) \cos. a\,;$$

or

$$\cos. A = \text{cotan. } h \text{ tang. } b,$$

$$\cos. A = \sin. B \cos. a\,;$$

which are the same as (289) and (296).

In like manner, if co. B were made the middle part, we should have

$$\cos. B = \text{cotan. } h \text{ tang. } a,$$

$$\cos. B = \sin. A \cos. b\,;$$

which are the same as (290) and (297).

Thirdly. If a were made the middle part, then co. B and b would be the adjacent parts, and co. A and co. h the opposite parts; and we should have

$$\sin. a = \text{tang. } (\text{co. } B) \text{ tang. } b.$$

$$\sin. a = \cos. (\text{co. } A) \cos. (\text{co. } h)\,;$$

or

$$\sin. a = \text{cotan. } B \text{ tang. } b,$$

$$\sin. a = \sin. A \sin. h\,;$$

which are the same as (294) and (291).

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In like manner, if b were made the middle part, we should have

$$\sin. b = \text{cotan.} A \text{ tang.} a,$$

$$\sin. b = \sin. B \sin. h;$$

which are the same as (293) and (292).

Having thus made each part successively the middle part, the ten equations, which we have obtained, must be all the equations included in Napier's Rules; and we perceive that they are identical with the ten equations [288–297].

12. *Theorem. The three sides of a spherical right triangle are either all less than* 90°; *or else, one is less while the other two are greater than* 90°, unless one of them is equal to 90°, as in § 16.

Proof. When h is less than 90°, the first member of (288) is positive; and therefore the factors of its second member must either be both positive or both negative; that is, the two legs a and b must, by Pl. Trig. § 62, be both acute or both obtuse.

But when h is obtuse, the first member of (288) is negative; and therefore one of the factors of the second member must be positive, while the other is negative; that is, of the two legs a and b, one must be acute, while the other is obtuse.

13. *Theorem. The hypothenuse differs less from* 90° *than does either of the legs,* the case of either side equal to 90° being excepted.

Proof. The factors cos. a and cos. b of the second member of the equation (288) are, by (4), fractions whose numerators are less than their denominators. Their product, neglecting the sines, must then be less than either of them, as cos. a for instance, or

$$\cos. h < \cos. a;$$

and therefore, by Pl. Trig. § 70 and 71, h must differ less from 90° than a does.

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14. *Theorem. An angle and its opposite leg in a spherical right triangle must be both acute, or both obtuse, or, by* § 16, *both equal to* 90°.

Proof. When A is acute, the first member of (296) is positive, and therefore the factor cos. a of the second member, being multiplied by the positive factor sin. B must be positive; that is, a must be acute. But if A is obtuse, the first member of (296) is negative, and therefore the factor cos. a of the second member must be negative; that is, a must be obtuse.

15. *Theorem. An angle differs less from* 90° *than its opposite leg*, the case of either side, equal to 90°, being excepted.

Proof. Since the second member of (296) is the product of the two fractions cos. a and sin. B, the first member must be less than either of them. Thus, neglecting the sines,

$$\cos. A < \cos. a;$$

hence A differs less from 90° than a does.

16. *Theorem. When in a spherical right triangle either side is equal to* 90°, *one of the other two sides is also equal to* 90°; *and each side is equal to its opposite angle.*

Proof. First. If either of the legs is equal to 90°, the corresponding factor of the second member of (288) is, by (66), equal to zero; which gives

$$\cos. h = 0,$$

or, by (66),

$$h = 90°.$$

Again, if we have

$$h = 90°,$$

it follows, from (66) and (288), that

$$0 = \cos. a \cos. b,$$

and therefore either cos. a or cos. b must be zero; that is, either a or b must be equal to 90°.

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Secondly. When either side is equal to 90°, it follows, from the preceding proof, that

$$h = 90°;$$

which substituted in (291) produces, by (67),

$$\sin. a = \sin. A;$$

which gives

$$a = A;$$

because, from § 14, a could not be equal to the supplement of A.

17. *Corollary.* When both the legs of a spherical right triangle are equal to 99°, all the sides and angles are equal to 90°.

18. *Theorem. When two of the angles of a spherical triangle are equal to* 90°, *the opposite sides are also equal to* 90°.

Proof. For in this case, one of the factors of the second member of the equation (295) must, by (68), be equal to zero, since either A or B is 90°; hence

$$\cos. h = 0;$$

or, by (66),

$$h = 90°;$$

and the remainder of the proposition follows from § 16.

19. *Corollary.* When all the angles of a spherical right triangle are equal to 90°, all the sides are also equal to 90°.

20. *Theorem. The sum of the angles of a spherical right triangle is greater than* 180°, *and less than* 360°; *and each angle is less than the sum of the other two.*

Proof. I. It is proved in Geometry, that the sum of the angles of any spherical triangle is greater than 180°.

II. It is proved in Geometry, that each angle of any spherical triangle is greater than the difference between two right angles

and the sum of the other two angles. Hence, if the sum of the two angles A and B is greater than 180°, we have

$$90° > A + B - 180°,$$

or $$A + B < 270°,$$

or $$A + B + 90° < 360°;$$

that is, the sum of the three angles is less than 360°; and in case the sum of the angles A and B is less than 180°, the sum of the three angles is obviously less than 360°.

III. When the right angle is greatest of the three angles, we have

$$90° + A + B > 180°,$$

or $$A + B > 90°;$$

that is, the greater angle is in this case less than the sum of the other two.

But if one of the other angles A is the greatest of the three angles, we have, by the proposition of Geometry last referred to,

$$B > 90° + A - 180°,$$

or $$B > A - 90°,$$

or $$A < B + 90°;$$

so that in every case one angle is less than the sum of the other two.

21. To solve a spherical right triangle, two parts must be known in addition to the right angle. From the two known parts, the other three parts are to be determined, separately, by equations derived from Napier's Rules. The two given parts, with the one to be determined, are, in each case, to enter into the same equation. *These three parts are either all adjacent to each other, in which case the middle one is taken as the* MIDDLE PART, *and the other two are, by* § 11, ADJACENT PARTS; *or one is separated from the other two, and then the part, which stands by itself, is the* MIDDLE PART, *and the other two are, by* § 11, OPPOSITE PARTS.

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22. *Problem. To solve a spherical right triangle, when the hypothenuse and one of the angles are given.*

Solution. Let ABC (fig. 30) be the right trangle, right-angled at C; and let the sides be denoted as in § 10. Let h and A be given; to solve the triangle.

First. To find the other angle B. The three parts, which are to enter into the same equation, are co. h, co. A, and co. B; and, by § 21, as they are all adjacent to each other, co. h is the middle part, and co. A and co. B are adjacent parts. Hence, by Napier's Rules,

$$\text{sin. (co. } h) = \text{tang. (co. } A) \text{ tang. (co. } B),$$

or

$$\text{cos. } h = \text{cotan. } A \text{ cotan. } B;$$

and, by (6),

$$\text{cotan. } B = \frac{\text{cos. } h}{\text{cotan. } A} = \text{cos. } h \text{ tang. } A.$$

Secondly. To find the opposite leg a. The three parts are co. A, co. h, and a; of which, by § 21, a is the middle part, and co. h and co. A are the opposite parts. Hence, by Napier's Rules,

$$\text{sin. } a = \text{cos. (co. } h) \text{ cos. (co. } A),$$

or

$$\text{sin. } a = \text{sin. } h \text{ sin. } A.$$

Thirdly. To find the adjacent leg. b. The three parts are co. A, co. h, and b; of which co. A is the middle part, and co. h and b are the adjacent parts. Hence, by Napier's Rules,

$$\text{sin. (co. } A) = \text{tang. (co. } h) \text{ tang. } b,$$

or

$$\text{cos. } A = \text{cotan. } h \text{ tang. } b;$$

and, by (6),

$$\text{tang. } b = \frac{\text{cos. } A}{\text{cotan. } h} = \text{tang. } h \text{ cos. } A.$$

23. *Scholium.* The tables always give two angles, which are supplements of each other, corresponding to each sine, cosine, &c. But it is easy to choose the proper angle for the particular case, by referring to § 12 and 14; or by having regard to the signs of the different terms of the equation, as determined by Pl. Trig. § 62.

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24. *Scholium.* When h and A are both equal to $90°$, the values of cotan. B and tang. b are indeterminate; for the numerators and denominators of the fractional values are, by (66) and (68), equal to zero; and in this case there are an infinite number of triangles which satisfy the given values of h and A.

The problem is impossible by § 18, if the given value of h differs from $90°$, while that of A is equal to $90°$.

25. Examples.

1. Given in the spherical right triangle (fig. 30), $h = 145°$ and $A = 23° 28'$; to solve the triangle.

Solution.

h, cos. 9.91336 n,*	sin. 9.75859,	tang. 9.84523 n
A, tang. 9.63761,	sin. 9.60012,	cos. 9.96251
B, cotan. 9.55097 n;	a sin. 9.35871;	b tang. 9.80774 n

Ans. $B = 109° 34' 33''$, $a = 13° 12' 12''$, $b = 147° 17' 15''$.

2. Given in the spherical right triangle (fig. 30), $h = 32° 34'$, and $A = 44° 44'$; to solve the triangle.

Ans. $B = 50° \ 8' 21''$,
$a = 22° 15' 43''$,
$b = 24° 24' 19''$.

26. *Problem. To solve a spherical triangle, when its hypothenuse and one of its legs are given.*

Solution. Let ABC (fig. 30) be the triangle, h the given hypothenuse, and a the given leg.

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* The letter n, placed after a logarithm, indicates it to be the logarithm of a negative quantity: and it is plain that, when the number of such logarithms to be added together is even, the sum is the logarithm of a positive quantity; but if odd, the sum is the logarithm of a negative quantity.

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First. To find the opposite angle A; a is the middle part, and co. A and co. h are the opposite parts. Hence

$$\sin. a = \cos. (\text{co. } h) \cos. (\text{co. } A);$$

or
$$\sin. a = \sin. h \sin. A;$$

and, by (6),

$$\sin. A = \frac{\sin. a}{\sin. h} = \sin. a \text{ cosec. } h.$$

Secondly. To find the adjacent angle B; co. B is the middle part, and co. h and a are the adjacent parts. Hence

$$\sin. (\text{co. } B) = \text{tang. } a \text{ tang. } (\text{co. } h),$$

or
$$\cos. B = \text{tang. } a \text{ cotan. } h.$$

Thirdly. To find the other leg b; co. h is the middle part, and a and b are the opposite parts. Hence

$$\cos. h = \cos. a \cos. b;$$

and, by (6),

$$\cos. b = \frac{\cos. h}{\cos. a} = \sec. a \cos. h.$$

27. *Scholium.* The question is impossible by § 13, when the given value of the hypothenuse differs more from 90° than that of the leg.

28. *Solution.* When h and a are both equal to 90°, it may be shown, as in § 24, that the values of B and b are indeterminate.

29. Example.

Given in the spherical right triangle (fig. 30), $a = 141° 11$, and $h = 127° 12'$; to solve the triangle.

Ans. $A = 128° \; 5' 54''$,

$B = 52° 21' 45''$,

$b = 39° \; 6' 23''$.

30. *Problem.* ***To solve a spherical right triangle, when one of its legs and the opposite angle are given.***

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Solution. Let ABC (fig. 30) be the triangle, a the given leg, and A the given angle.

First. To find the hypothenuse h; a is the middle part, and co. h and co. A are the opposite parts. Hence

$$\sin. a = \sin. h \sin. A;$$

and, by (6),

$$\sin. h = \frac{\sin. a}{\sin. A} = \sin. a \operatorname{cosec}. A.$$

Secondly. To find the other angle B; co. A is the middle part, and a and co. B are the opposite parts. Hence

$$\cos. A = \cos. a \sin. B;$$

and, by (6),

$$\sin. B = \frac{\cos. A}{\cos. a} = \sec. a \cos. A.$$

Thirdly. To find the other leg b; b is the middle part, and a and co. A are the adjacent parts. Hence

$$\sin. b = \operatorname{tang}. a \operatorname{cotan}. A.$$

31. *Scholium.* There are two triangles ABC and $A'BC$ (fig. 31) formed by producing the sides AB and AC, to the point of meeting A', both of which satisfy the conditions of the problem. For the side BC or a, and the angle A, or by § 2 its equal A', belong to both the triangles.

Now ABA' and ACA' are semicircumferences. Hence h', the hypothenuse of $A'BC$, is the supplement of h; b' is the supplement of b; and $A'BC$, is the supplement of ABC. One set of values, then, of the unknown quantities, given by the tables, as in § 23, corresponds to the triangle ABC, and the other set to $A'BC$.

32. *Corollary.* When the given values of a and A are equal, the values of h, B, and b become

$$\sin. h = 1, \ \sin. B = 1, \ \sin. b = 1;$$

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or, by (67),

$$h = 90°, \quad B = 90°, \quad b = 90°;$$

as in § 16.

33. *Corollary.* When a and A are equal to $90°$, the values of b and B are indeterminate, as in § 24.

34. *Scholium.* The problem is, by § 14, impossible, when the given values of the leg and its opposite angle are such, that one is obtuse, while the other is acute, or that one is equal to $90°$, while the other differs from $90°$; and, by § 15, it is impossible, when the given value of the angle differs more from $90°$ than that of the leg.

35. Example.

Given in the spherical right triangle, (fig. 30), $a = 35° \, 44'$, and $A = 37° \, 28'$; to solve the triangle.

$$\textit{Ans.} \quad \left.\begin{aligned} h &= 73° \, 45' \, 15'' \\ B &= 77° \, 54' \\ b &= 69° \, 50' \, 24'' \end{aligned}\right\} \text{ or } \left\{\begin{aligned} h &= 106° \, 14' \, 45'' \\ B &= 102° \quad 6' \\ b &= 110° \quad 9' \, 36''. \end{aligned}\right.$$

36. *Problem.* ***To solve a spherical right triangle, when one of its legs and the adjacent angle are given.***

Solution. Let ABC (fig. 30) be the triangle, a the given leg, and B the given angle.

First. To find the hypothenuse h; co. B is the middle part, and co. h and a are adjacent parts. Hence

$$\cos. B = \text{tang. } a \text{ cotan. } h;$$

and, by (6),

$$\text{cotan. } h = \frac{\cos. B}{\text{tang. } a} = \text{cotan. } a \cos. B.$$

Secondly. To find the other angle A; co. A is the middle part, and co. B and a are opposite parts. Hence

$$\cos. A = \cos. a \sin. B.$$

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Thirdly. To find the other leg b; a is the middle part, and co. B and b are adjacent parts. Hence

$$\sin. a = \text{tang. } b \text{ cotan. } B;$$

and, by (6),

$$\text{tang. } b = \frac{\sin. a}{\text{cotan. } B} = \sin. a \text{ tang. } B.$$

37. Example.

Given in the spherical right triangle (fig. 30), $a = 118° 54'$, and $B = 12° 19'$; to solve the triangle.

Ans. $h = 118° 20' 20''$,
$A = 95° 55' 2''$,
$b = 10° 49' 17''$.

38. *Problem. To solve a spherical right triangle, when its two legs are given.*

Solution. Let ABC (fig. 30) be the triangle, a and b the given legs.

First. To find the hypothenuse h; co. h is the middle part, a and b are opposite parts. Hence

$$\cos. h = \cos. a \cos. b$$

Secondly. To find one of the angles, as A; b is the middle part, and co. A and a are adjacent parts. Hence

$$\sin. b = \text{tang. } a \text{ cotan. } A;$$

and, by (6),

$$\text{cotan. } A = \frac{\sin. b}{\text{tang. } a} = \text{cotan. } a \sin. b.$$

In the same way,

$$\text{cotan. } B = \text{cotan. } b \sin. a.$$

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39. Example.

Given in the spherical right triangle (fig. 30), $a = 1°$, and $b = 100°$; to solve the triangle.

$$\textit{Ans.} \quad h = 99° \, 59' \, 54'', \quad A = 1° \, 0' \, 56'', \quad B = 90° \, 10' \, 35''.$$

40. *Problem. To solve a spherical right triangle, when the two angles are given.*

Solution. Let ABC (fig. 30) be the triangle, A and B the given angles.

First. To find the hypothenuse h; co. h is the middle part, and co. A and co. B are adjacent parts. Hence

$$\cos. h = \text{cotan.}\, A \,\text{cotan.}\, B.$$

Secondly. To find one of the legs, as a; co. A is the middle part, and co. B and a are the opposite parts. Hence

$$\cos. A = \cos. a \sin. B;$$

and, by (6),

$$\cos. a = \frac{\cos. A}{\sin. B} = \cos. A \,\text{cosec.}\, B.$$

In the same way,

$$\cos. b = \text{cosec.}\, A \cos. B.$$

41. *Scholium.* The problem is, by § 20, impossible, when the sum of the given values of A and B is less than 90°, or greater than 270°, or when their difference is greater than 90°.

42. Example.

Given in the spherical right triangle (fig. 30), $A = 91° \, 11'$, and $B = 111° \, 11'$; to solve the triangle.

$$\textit{Ans.} \quad h = 89° \, 32' \, 29'', \quad a = 91° \, 16' \, 8'', \quad b = 111° \, 11' \, 16''.$$

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CHAPTER III.

SPHERICAL OBLIQUE TRIANGLES.

43. *Theorem. The sines of the sides in any spherical triangle, are proportional to the sines of the opposite angles.* [B. p. 437.]

Proof. Let ABC (figs. 32 and 33) be the given triangle. Denote by a, b, c, the sides respectively opposite to the angles A, B, C. From either of the vertices let fall the perpendicular BP upon the opposite side AC. Then, in the right triangle ABP, making BP the middle part, co. c and co. BAP are the opposite parts. Hence, by Napier's Rules,

$$\sin. BP = \sin. c \sin. BAP = \sin. c \sin A.$$

For BAP is either the same as A, or it is its supplement, and in either case has the same sine, by (98).

Again, in triangle BPC, making BP the middle part, co. a and co. C are the opposite parts. Hence, by Napier's Rules,

$$\sin. BP = \sin. a \sin. C;$$

and, from the two preceding equations,

$$\sin. c \sin. A = \sin a \sin C,$$

which may be written as a proportion, as follows:

$$\sin. a : \sin. A = \sin. c : \sin. C.$$

In the same way,

$$\sin. a : \sin. A = \sin. b : \sin. B.$$

44. *Theorem. Bowditch's Rules for Oblique Triangles.* If, in a spherical triangle, two right triangles are formed by a perpendicular let fall from one of its verticles upon the oppo-

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site side; and if, in the two right triangles, the middle parts are so taken that the perpendicular is an adjacent part in both of them; then

The sines of the middle parts in the two triangles are proportional to the tangents of the adjacent parts.

But, if the perpendicular is an opposite part in both the triangles, then

The sines of the middle parts are proportional to the cosines of the opposite parts. [B. p. 437.]

Proof. Let M denote the middle part in one of the right triangles, A an adjacent part, and O an opposite part. Also let m denote the middle part in the other right triangle, a an adjacent part, and o an opposite part, and let p denote the perpendicular.

First. If the perpendicular is an adjacent part in both triangles, we have, by Napier's Rules,

$$\sin. M = \text{tang. } A \text{ tang. } p,$$

$$\sin. m = \text{tang. } a \text{ tang. } p;$$

whence

$$\frac{\sin. M}{\sin. m} = \frac{\text{tang. } A \text{ tang. } p}{\text{tang. } a \text{ tang. } p} = \frac{\text{tang. } A}{\text{tang. } a},$$

or $$\sin. M : \sin. m = \text{tang. } A : \text{tang. } a.$$

Secondly. If the perpendicular is an opposite part in both the triangles, we have, by Napier's Rules,

$$\sin. M = \cos. O \cos. p,$$

$$\sin. m = \cos. o \cos. p;$$

whence

$$\frac{\sin. M}{\sin. m} = \frac{\cos. O \cos. p}{\cos. o \cos. p} = \frac{\cos. O}{\cos. o},$$

or $$\sin. M : \sin. m = \cos. O : \cos. o.$$

45. *Problem. To solve a spherical triangle, when two of its sides and the included angle are given.* [B. p. 438.]

Solution. Let ABC (figs. 32 and 33) be the triangle, a and b the given sides, and C the given angle. From B let fall on AC the perpendicular BP.

Frst. To find PC, we know, in the right triangle BPC, the hypothenuse a and the angle C. Hence, by means of Napier's Rules,

$$\text{tang. } PC = \cos. C \text{ tang. } a. \tag{298}$$

Secondly. AP is the difference between AC and PC, that is,

$$\text{(fig. 32) } AP = b - PC, \text{ or (fig. 33) } AP = PC - b. \tag{299}$$

Thirdly. To find the side c. If, in the triangle BPC, co. a is the middle part, PC and PB are opposite parts; and if, in the triangle APB, co. c is the middle part, BP and AP are the opposite parts. Hence, by Bowditch's Rules,

$$\cos. PC : \cos. AP = \sin. (\text{co. } a) : \sin. (\text{co. } c),$$

or
$$\cos. PC : \cos. AP = \cos. a : \cos. c. \tag{300}$$

Fourthly. To find the angle A. If, in the triangle BPC, PC is the middle part, co. C and BP are adjacent parts; and if, in the triangle APB, AP is the middle part, co. BAP and BP are adjacent parts. Hence, by Bowditch's Rules,

$$\sin. PC : \sin. PA = \text{cotan. } C : \text{cotan. } BAP; \tag{301}$$

and BAP is the angle A (fig. 32), when the perpendicular falls within the triangle; or it is the supplement of A (fig. 33), when the perpendicular falls without the triangle.

Fifthly. B is found by means of § 43,

$$\sin. c : \sin. C = \sin. b : \sin. B. \tag{302}$$

46. *Scholium.* In determining PC, c, and BAP, by (298), (300), and (301), the signs of the several terms must be carefully attended to; by means of Pl. Trig. § 62.

But to determine which value of B, determined by (302), is the true value, regard must be had to the following rules, which are proved in Geometry.

I. *The greater side of a spherical triangle is always opposite to the greater angle.*

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II. *Each side is less than the sum of the other two.*

III. *The sum of the sides is less than* 360°.

IV. *Each angle is greater than the difference between* 180°, *and the sum of the other two angles.*

There are, however, cases in which these conditions are all satisfied by each of the values of B. In any such case this angle can be determined in the same way in which the angle A was determined, by letting fall a perpendicular from the vertex A on the side BC. But this difficulty can always be avoided, by letting fall the perpendicular upon that of the two given sides which differs the most from 90°.

47. *Corollary.* By (299), (111), and (35), we have

$$\cos. AP = \cos. (b - PC) = \cos. (PC - b)$$

$$= \cos. b \cos. PC + \sin. b \sin. PC, \tag{303}$$

which, substituted in (300), gives

$$\cos. PC : \cos. b \cos. PC + \sin. b \sin. PC = \cos. a : \cos. c.$$

Dividing the two terms of the first ratio by cos. PC, we have by (7),

$$1 : \cos. b + \sin. b \operatorname{tang.} PC = \cos. a : \cos. c. \tag{304}$$

The product of the means being equal to that of the extremes, we have

$$\cos. c = \cos. a \cos. b + \sin. b \cos. a \operatorname{tang.} PC. \tag{305}$$

But, by (298),

$$\operatorname{tang.} PC = \cos. C \operatorname{tang.} a = \frac{\cos. C \sin. a}{\cos. a},$$

or

$$\cos. a \operatorname{tang.} PC = \cos. C \sin. a; \tag{306}$$

which, substituted in (305), gives

$$\cos. c = \cos. a \cos. b + \sin. a \sin. b \cos. C, \tag{307}$$

which is *one of the fundamental equations of Spherical Trigonometry.*

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48. *Corollary.* We have, by (55),

$$\cos. C = -1 + 2 (\cos. \tfrac{1}{2} C)^2,$$

which, substituted in (307), gives, by (34),

$$\cos. c = \cos. (a + b) + 2 \sin. a \sin. b (\cos. \tfrac{1}{2} C)^2, \qquad (308)$$

from which the value of the side c can readily be found by using the table of Natural Sines.

49. *Corollary.* We have, by (56),

$$\cos. C = 1 - 2 (\sin. \tfrac{1}{2} C)^2,$$

which, substituted in (307), gives, by (35),

$$\cos. c = \cos. (a - b) - 2 \sin. a \sin. b (\sin. \tfrac{1}{2} C)^2, \qquad (309)$$

which can be used like formula (308).

50. *Corollary.* The use of formula (309) is much facilitated by means of the column of Rising in Table XXIII of the Navigator. This column contains the values of

$$\begin{aligned} \log. 2 (\sin. \tfrac{1}{2} C)^2 &= 2 \log. \sin. \tfrac{1}{2} C + \log. 2 \\ &= 2 \log. \sin. \tfrac{1}{2} C + 0.30103. \qquad (310) \end{aligned}$$

But the decimal point is supposed to be changed so as to correspond to the table of Natural Sines, that is, 5 is added to the logarithm; and 20 is to be subtracted from the value of 2 log. sin. $\frac{1}{2}$ C, which is given by Table XXVII, as is evident from Pl. Trig. § 31. So that the column Rising of Table XXIII is constructed by the formula

$$\begin{aligned} \log. \text{Ris. } C &= 2 \log. \sin. \tfrac{1}{2} C + 5.30102 - 20 \\ &= 2 \log. \sin. \tfrac{1}{2} C - 14.69897, \qquad (311) \end{aligned}$$

which agrees with the explanation in the Preface to the Navigator.

51. By using Table XXIII, the following rule is obtained for finding the third side, when two sides and the included angle are given.

Add together the log. Rising of the given angle, and the log. sines of the two given sides. The sum is the logarithm of a

number, which is to be subtracted from the natural cosine of the difference of two given sides (regard being had to the sign of this cosine). The difference is the natural cosine of the required side.

52. Examples.

1. Calculate the value of log. Ris. of $4^h\,28^m$.

Solution.

$\frac{1}{2}(4^h\,28^m) = 2^h\,14^m$	sin.	9.74189
		2
		19.48378
		— 14.69897
log. Ris. $4^h\,28^m$		= 4.78481

2. Given in the spherical triangle two sides equal to 45° 54′ and 138° 32′, and the included angle 98° 44′; to solve the triangle.

Solution. I. By (298),

$C =$ 98° 44′	cos.	9.18137 n
$a =$ 45° 54′	tang.	0.01365
$PC =$ 171° 5′ 43″	tang.	9.19502 n

By (299),

$$AP = 171°\ 5'\ 43'' - 138°\ 32' = 32°\ 33'\ 43''.$$

By (300),

$PC =$ 171° 5′ 43″	cos.	(ar. co.)	10.00527 n
$AP =$ 32° 33′ 43″	cos.		9.92573
$a =$ 45° 54′	cos.		9.84255
$c =$ 126° 25′ 7″	cos.		9.77355 n

By (301),

$PC = 171^\circ\ 5'\ 43''$	sin.	(ar. co.)	10.81024
$AP = 32^\circ\ 33'\ 43''$	sin.		9.73095
$C = 98^\circ\ 44'$	cotan.		9.18644_n
$BAP = 118^\circ\ 6'\ 26''$	cotan.		9.72763_n

$$A = 180^\circ - 118^\circ\ 6'\ 26'' = 61^\circ\ 53'\ 34''.$$

By (302),

$c = 126^\circ\ 25'\ 7''$	sin.	(ar. co.)	10.09436
$C = 98^\circ\ 44'$	sin.		9.99494
$b = 138^\circ\ 32'$	sin.		9.82098
$B = 125^\circ\ 34'\ 30''$	sin.		9.91028

Ans. $c = 126^\circ\ 25'\ 7''$

$A = 61^\circ\ 53'\ 34''$

$B = 125^\circ\ 34'\ 30''$.

II. The third side is thus calculated by means of (308),

2	log.	0.30103
$45^\circ\ 54'$	sin.	9.85620
$138^\circ\ 32'$	sin.	9.82098
$\frac{1}{2}(98^\circ\ 44') = 49^\circ\ 22'$	2 cos.	19.62744
0.40332		9.60565

$-\ 0.99701 =$ Nat. cos. $(138^\circ\ 32' + 45^\circ\ 54') =$ N. cos. $184^\circ\ 26'$

$-\ 0.59369 =$ Nat. cos. $126^\circ\ 25'\ 10'' = c$.

III. The third side is thus calculated by § 50.

$98^\circ\ 44' = 6^h\ 34^m\ 56^s$		log. Ris.	5.06139
$45^\circ\ 54'$		sin.	9.85620
$138^\circ\ 32'$		sin.	9.82098
	54774		4.73857
$92^\circ\ 38'$ N. cos.	$-$ 4594		
$c = 126^\circ\ 25'\ 8''$ N. cos.	$-$ 59368		

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3. Calculate the log. Ris. of $11^h\ 12^m\ 20^s$. *Ans.* 5.29632.

4. Given in a spherical triangle two sides equal to 100°, and 125°, and the included angle equal to 45°; to solve the triangle.

Ans. The third side $= 47°\ 55'\ 52''$.

The other two angles $= 69°\ 43'\ 48''$, and $= 128°\ 42'\ 48''$.

53. *Problem. To solve a spherical triangle, when one of its sides and the two adjacent angles are given.* [B. p. 438.]

Solution. Let ABC (figs. 32 and 33) be the triangle, a the given side, and B and C the given angles. From B let fall on AC the perpendicular BP.

First. To find PBC. We know, in the right triangle BPC, the hypothenuse a and the angle C. Hence, by Napier's Rules,

$$\text{cotan. } PBC = \cos.\ a \text{ tang. } C. \tag{312}$$

Secondly. ABP is the difference between ABC and PBC, that is,

$$\text{(fig. 32) } ABP = B - PBC,$$

or

$$\text{(fig. 33) } ABP = PBC - B. \tag{313}$$

Thirdly. To find the angle A. If, in the triangle PBC, co. C is the middle part, PB and co. PBC are the opposite parts; and if, in the triangle ABP, co. BAP is the middle part, PB and co. ABP are the opposite parts. Hence, by Bowditch's Rules,

$$\cos.\ (\text{co. } BPC) : \cos.\ (\text{co. } ABP) = \sin.\ (\text{co. } C) : \sin.\ (\text{co. } BAP),$$

or

$$\sin.\ PBC : \sin.\ ABP = \cos.\ C : \cos.\ BAP; \tag{314}$$

and BAP is either the angle A or its supplement.

Fourthly. To find the side c. If, in the triangle PBC, co. PBC is the middle part, PB and co. a are the adjacent parts; and if, in the triangle ABP, co. ABP is the middle part, PB and co. c are the adjacent parts. Hence, by Bowditch's Rules,

$$\cos.\ PBC : \cos.\ ABP = \text{cotan. } a : \text{cotan. } c. \tag{315}$$

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Fifthly. b is found by the proportion

$$\sin. C : \sin. c = \sin. B : \sin. b. \tag{316}$$

54. *Scholium.* In determining PBC, BAP, and c by (312), (314), and (315), the signs of the several terms must be carefully attended to, by means of Pl. Trig. § 61.

To determine which value of b, obtained from (316), is the true value, regard must be had to the rules of § 46. But if all these conditions are satisfied by both values of b, then b may be calculated by letting fall a perpendicular from C on the side c in the same way in which c has been obtained in the preceding solution. But this case can be avoided by letting fall the perpendicular from the vertex of that one of the two given angles, which differs the most from 90°.

55. *Corollary.* Since $180° - a$, $180° - b$, and $180° - c$ are the angles of the polar triangle, and $180° - A$, $180° - B$, and $180° - C$ are its sides; we have given in the polar triangle the two sides $180° - B$, and $180° - C$, and the included $180° - a$; so that the polar triangle might be solved by § 45.

56. *Corollary.* If formula (307) is applied to the polar triangle of the preceding section, it becomes by Pl. Trig. § 61,

$$- \cos. A = \cos. B \cos. C - \sin. B \sin. C \cos. a,$$

or $$\cos. A = - \cos. B \cos. C + \sin. B \sin. C \cos. a. \tag{317}$$

57. *Corollary.* In the same way (308) becomes by (99) and (123),

$$\cos. A = - \cos. (B + C) - 2 \sin. B \sin. C (\sin. \tfrac{1}{2} a)^2, \tag{318}$$

from which the value of the third angle may be found by means of Table XXIII.

58. *Corollary.* In the same way (309) becomes by (99),

$$\cos. A = - \cos. (B - C) + 2 \sin. B \sin. C (\cos. \tfrac{1}{2} a)^2, \tag{319}$$

from which the value of the third side may be found.

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59. Examples.

1. Given in a spherical triangle one side equal to 175° 27′, and the two adjacent angles equal to 126° 12′, and 109° 16′; to solve the triangle.

Solution. I. By (312),

$a = 175° 27'$	cos.	9.99863 n
$C = 109° 16'$	tang.	0.45650 n
$PBC = 19° 19' 24''$	cotan.	0.45513

By (313),

$$ABP = 126° 12' - 19° 19' 24'' = 106° 52' 36''.$$

By (314),

$PBC = 19° 19' 24''$	sin.	(ar. co.)	10.48031
$ABP = 106° 52' 36''$	sin.		9.98088
$C = 109° 16'$	cos.		9.51847 n
$BAP = 162° 36'$	cos.		9.97966 n

By (315),

$PBC = 19° 19' 24''$	cos.	(ar. co.)	10.02518
$ABP = 106° 52' 36''$	cos.		9.46288 n
$a = 175° 27'$	cotan.		1.09920 n
$c = 14° 30' 9''$	cotan.		0.58726

$$A = BAP = 162° 36'.$$

By (316),

$C = 109° 16'$	sin.	(ar. co.)	10.02503
$c = 14° 30' 9''$	sin.		9.39867
$B = 126° 12'$	sin.		9.90685
$b = 167° 38' 21''$	sin.		9.33055

Ans. $A = 162° 36'$

$b = 167° 38' 21''$

$c = 14° 30' 9''$.

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II. The third angle is thus calculated by means of (318).

$175^\circ 27' = 11^h 41^m 48^s$		log. Ris.	5.30035
$126^\circ 12'$		sin.	9.90685
$109^\circ 16'$		sin.	9.97497
	— 152114		5.18217
$235^\circ 28'$ — N. cos.	56689		
$A = 162^\circ 36' 7''$ N. cos.	— 95425		

III. The third angle is thus calculated by means of (319),

		2 log.	0.30103
$\frac{1}{2}(175^\circ 27') = 87^\circ 43' 30''$		2 cos.	17.19748
$126^\circ 12'$		sin.	9.90685
$109^\circ 16'$		sin.	9.97497
	0.00240		7.38033
$16^\circ 56'$ — N. cos.	— 0.95664		
$A = 162^\circ 36'$ N. cos.	— 0.95424		

2. Given in a spherical triangle one side $= 45^\circ 54'$, and the two adjacent angles $= 125^\circ 37'$, and $= 98^\circ 44'$; to solve the triangle.

Ans. The third triangle $= 61^\circ 55' 2''$.

The other two sides $= 138^\circ 34' 22''$, and $= 126^\circ 26' 11''$.

60. *Problem. To solve a spherical triangle, when two sides and an angle opposite one of them are given.* [B. p. 437.]

Solution. Let ABC (figs. 32 and 33) be the triangle, a and c the given sides, and C the given angle. From B let fall on AC the perpendicular BP.

First. To find PC. We know, in the right triangle PBC, the side a and the angle C. Hence, by Napier's Rules,

$$\text{tang. } PC = \cos. C \text{ tang. } a. \qquad (320)$$

Secondly. To find AP. If, in the triangle PBC, co. a is the middle part, CP and PB are the opposite parts; and if, in the tri-

angle ABP, co. c is the middle part, AP and PB are the opposite parts. Hence, by Bowditch's Rules,

$$\cos. a : \cos. c = \cos. PC : \cos. AP. \tag{321}$$

Thirdly. To find b. There are, in general, two triangles which resolve the problem, in one of which (fig. 32)

$$b = PC + AP, \tag{322}$$

and in the other (fig. 33)

$$b = PC - AP. \tag{323}$$

But, if AP is greater than PC, there is but one triangle, as in (fig. 32), and b is obtained by (322); or, if the sum of AP and PC is greater than 180°, there is but one triangle, as in (fig. 33), and b is obtained by (323).

Fourthly. A and B are found by the proportion

$$\sin. c : \sin. C = \sin. a : \sin. A \tag{324}$$

$$\sin. c : \sin. C = \sin. b : \sin. B. \tag{325}$$

61. *Scholium.* In determining PC and AP by (320) and (321), the signs of the several terms must be carefully attended to by means of Pl. Trig. § 62.

The two values of A, given by (324), correspond respectively to the two triangles which satisfy the problem; and the one, which belongs to each triangle, is to be selected, so that the angle BAP, which is the same as A in (fig. 32), and the supplement of A in (fig. 33), may be obtuse if C is obtuse, and acute if C is acute. For BP is the side opposite BAP in the right triangle ABP, and the side opposite C in the triangle BCP; and therefore, by § 14, BP, BAP, and C are all at the same time acute, or all obtuse.

Of the two values of B, given by (325), the one which belongs to each triangle is to be determined by means of the rules of § 46.

62. *Scholium.* The problem is, by a proposition of Geometry, impossible, when the given value of c differs more from 90° than that of a; if, at the same time, the value of one of the two quantities, c and C, is acute, while that of the other is obtuse. And in

this case we should find that AP was larger than PC, and at the same time that the sum of AP and PC was more than 180°.

63. Examples.

1. Given in the spherical triangle, one side $= 35°$, a second side $= 142°$, the angle opposite the second side $= 176°$; to solve the triangle.

Solution. By (320),

$C = 176°$	cos.	9.99894_n
$a = 35°$	tang.	9.84523
$PC = 145°\ 3'\ 56''$	tang.	9.84417_n

By (321),

$a = 35°$	cos. (ar. co.)	10.08664
$PC = 145°\ 3'\ 56''$	cos.	9.91371_n
$c = 142°$	cos.	9.89653_n
$AP = 37°\ 56'\ 30''$	cos.	9.89688

By (323),

$$b = 145°\ 3'\ 56'' - 37°\ 56'\ 30'' = 107°\ 7'\ 26''.$$

By (324),

$c = 142°$	sin. (ar. co.)	10.21066
$C = 176°$	sin.	8.84358
$a = 35°$	sin.	9.75859
$A = 3°\ 43'\ 34''$	sin.	8.81283

By (325),

$c = 142°$	sin. (ar. co.)	10.21066
$C = 176°$	sin.	8.84358
$b = 107°\ 7'\ 26''$	sin.	9.98030
$B = 6°\ 12'\ 58''$	sin.	9.03454

Ans. $b = 107°\ 7'\ 26''$
$A = 3°\ 43'\ 34''$
$B = 6°\ 12'\ 58''$.

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2. Given in a spherical triangle, one side $= 54°$, a second side $= 22°$, the angle opposite the second side $= 12°$; to solve the triangle.

Ans. The third side $= 73° \ 14' \ 29''$, or $= 33° \ 32' \ 59''$.

One angle $= 26° \ 40' \ 49''$, or $= 153° \ 19' \ 11''$.

The third angle $= 147° \ 53' \ 51''$, or $= 17° \ 51' \ 43''$.

64. *Problem. To solve a spherical triangle, when two angles and a side opposite one of them are given.* [B. p. 438.]

Solution. Let ABC (figs. 32 and 33) be the triangle, A and C the given angles, and a the given side.

From B let fall on AC the perpendicular BP. This perpendicular must fall within the triangle, if A and C are either both obtuse or both acute; but it falls without, if one is obtuse and the other acute.

First. PC may be found by (320).

Secondly. To find AP. If, in the triangle PBC, PC is the middle part, co. C and PB are the adjacent parts; and if, in the triangle ABP, AP is the middle part, co. BAP and BP are the adjacent parts. Hence, by Bowditch's Rules,

$$\text{cotan. } C : \text{cotan. } BAP = \sin.\ PC : \sin.\ AP. \tag{326}$$

Thirdly. To find b. We have

$$\text{(fig. 32)} \qquad b = PC + AP, \tag{327}$$

$$\text{(fig. 33)} \qquad b = PC - AP. \tag{328}$$

Fourthly. c and B are found by the proportion

$$\sin.\ A : \sin.\ a = \sin.\ C : \sin.\ c, \tag{329}$$

$$\sin.\ a : \sin.\ A = \sin.\ b : \sin.\ B. \tag{330}$$

65. *Scholium.* Either value of AP, given by (326), may be used, and there will be two different triangles solving the problem, except when $AP + PC$ (fig. 32) is greater than $180°$, or PC (fig. 33) is less than AP. It may be that both values of AP satisfy the con-

ditions of the problem, or that only one value satisfies them, or that neither value does; in which last case the problem is impossible.

Of the values of c, determined by (329), the true value must be ascertained from the right triangle ABP by § 12; or since PB and C are both acute or both obtuse at the same time, it follows, from § 12, that when C and AP are both acute or both obtuse, that c is acute; but when one of them is obtuse and the other acute, c is obtuse.

From the two values of B (330), the true value must be selected by means of the rules of § 46.

66. *Scholium.* The problem is impossible, by Geometry, when A differs more from 90° than does C, and when at the same time one of the two quantities a and A is acute, while the other is obtuse. This case is precisely the same as the impossible case of § 62.

67. Examples.

1. Given in a spherical triangle, one angle = 95°, a second angle = 104°, and the side opposite the first angle = 138°; to solve the triangle.

Solution. By (320),

$C = 104°$	cos.	9.38368_n
$a = 138°$	tang.	9.95444_n
$PC = 12° 17' 20''$	tang.	9.33812

By (326),

$C = 104°$	cotan. (ar. co.)	0.60323_n
$PC = 12° 17' 20''$	sin.	9.32805
$BAP = 95°$	cotan.	8.94195_n
$AP = 4° 16' 59''$	sin.	8.87323

By (327),

$$b = 12° 17' 20'' + 4° 16' 59'' = 16° 34' 19''.$$

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By (329),

$A = 95°$	sin. (ar. co.)	10.00166
$a = 138°$	sin.	9.82551
$C = 104°$	sin.	9.98690
$c = 139° \; 19' \; 40''$	sin.	9.81407

By (330),

$a = 138°$	sin. (ar. co.)	10.17449
$A = 95°$	sin.	9.99834
$b = 16° \; 34' \; 19''$	sin.	9.45518
$B = 25° \; 7' \; 38''$	sin.	9.62801

Again, by (328),

$$b = 12° \; 17' \; 20'' - 4° \; 16' \; 59'' = 8° \; 0' \; 21''$$

$$c = 180° - 139° \; 19' \; 40'' = 40° \; 40' \; 20''.$$

By (330),

$a = 138°$	sin. (ar. co.)	10.17449
$A = 95°$	sin.	9.99834
$b = 8° \; 0' \; 21''$	sin.	9.14386
$B = 11° \; 58' \; 0''$	sin.	9.31669

Ans. $b = 16° \; 34' \; 19''$ or $= 8° \; 0' \; 21''$

$c = 139° \; 19' \; 40''$ or $= 40° \; 40' \; 20''$

$B = 25° \; 7' \; 38''$ or $= 11° \; 58' \; 0''$.

2. Given in a spherical triangle, one angle $= 104°$, a second angle $= 95°$, and the side opposite the first angle $= 138°$; to solve the triangle.

Ans. The two sides are $17° \; 22' \; 13''$, and $136° \; 36' \; 27''$.

The other angle is $25° \; 39' \; 9''$.

68. *Problem. To solve a spherical triangle, when its three sides are given.*

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Solution. Equation (307) gives, by transposition and division,

$$\cos. C = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}, \tag{331}$$

whence the value of the angle C may be calculated, and in the same way either of the other angles.

69. *Corollary.* An equation, more easy for calculation by logarithms, may be obtained from (308), which gives, by transposition and division,

$$2 (\cos. \tfrac{1}{2} C)^2 = \frac{\cos. c - \cos. (a + b)}{\sin. a \sin. b}. \tag{332}$$

Now, letting s denote half the sum of the sides, or

$$s = \tfrac{1}{2} (a + b + c); \tag{333}$$

if we make in (42)

$$M = \tfrac{1}{2} (a + b + c) = s,$$

$$N = \tfrac{1}{2} (a + b - c) = s - c;$$

we have

$$M + N = a + b,$$

$$M - N = c;$$

and (42) becomes

$$\cos. c - \cos. (a + b) = 2 \sin. s \sin. (s - c);$$

which, substituted in (332), gives

$$2 (\cos. \tfrac{1}{2} C)^2 = \frac{2 \sin. s \sin. (s - c)}{\sin. a \sin. b}, \tag{334}$$

and

$$\cos. \tfrac{1}{2} C = \sqrt{\left(\frac{\sin. s \sin. (s - c)}{\sin. a \sin. b}\right)}. \tag{335}$$

70. *Corollary.* The angles A and B may be found by the two following equations, which are easily deduced from (335),

$$\cos. \tfrac{1}{2} A = \sqrt{\left(\frac{\sin. s \sin. (s - a)}{\sin. b \sin. c}\right)}. \tag{336}$$

$$\cos. \tfrac{1}{2} B = \sqrt{\left(\frac{\sin. s \sin. (s - b)}{\sin. a \sin. c}\right)}. \tag{337}$$

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71. *Corollary.* Another equation, equally simple in calculation, can be obtained from (309), which gives, by transposition and division,

$$2\,(\sin.\tfrac{1}{2}\,C)^2 = \frac{\cos.\,(a-b)-\cos.\,c}{\sin.\,a\,\sin.\,b}, \tag{338}$$

whence C can be found by Table XXIII.

72. *Corollary.* If, in (42), we make

$$M = \tfrac{1}{2}\,(a-b+c) = s-b$$
$$N = \tfrac{1}{2}\,(-a+b+c) = s-a,$$

we have

$$M+N = c$$
$$M-N = a-b,$$

and (42) becomes

$$\cos.\,(a-b)-\cos.\,c = 2\sin.\,(s-a)\,\sin.\,(s-b)\,;$$

which, substituted in (338) gives

$$2\,(\sin.\tfrac{1}{2}\,C)^2 = \frac{2\sin.\,(s-a)\,\sin.\,(s-b)}{\sin.\,a\,\sin.\,b}, \tag{339}$$

and

$$\sin.\tfrac{1}{2}\,C = \sqrt{\left(\frac{\sin.\,(s-a)\,\sin.\,(s-b)}{\sin.\,a\,\sin.\,b}\right)}. \tag{340}$$

73. *Corollary.* In the same way we might deduce the following equations:

$$\sin.\tfrac{1}{2}\,A = \sqrt{\left(\frac{\sin.\,(s-b)\,\sin.\,(s-c)}{\sin.\,b\,\sin.\,c}\right)}. \tag{341}$$

$$\sin.\tfrac{1}{2}\,B = \sqrt{\left(\frac{\sin.\,(s-a)\,\sin.\,(s-c)}{\sin.\,a\,\sin.\,c}\right)}. \tag{342}$$

74. *Corollary.* The quotient of (341), divided by (336), is by (7),

$$\text{tang.}\,\tfrac{1}{2}\,A = \frac{\sin.\tfrac{1}{2}\,A}{\cos.\tfrac{1}{2}\,A} = \sqrt{\left(\frac{\sin.\,(s-b)\,\sin.\,(s-c)}{\sin.\,s\,\sin.\,(s-a)}\right)}. \tag{343}$$

In the same way,

$$\text{tang.}\,\tfrac{1}{2}\,B = \sqrt{\left(\frac{\sin.\,(s-a)\,\sin.\,(s-c)}{\sin.\,s\,\sin.\,(s-b)}\right)}. \tag{344}$$

$$\text{tang.}\,\tfrac{1}{2}\,C = \sqrt{\left(\frac{\sin.\,(s-a)\,\sin.\,(s-b)}{\sin.\,s\,\sin.\,(s-c)}\right)}. \tag{345}$$

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75. Examples.

I. Given in the spherical triangle ABC the three sides equal to 46°, 72°, and 68°; to solve the triangle.

Solution.	I. By (335),	by (336),	by (337),
$a=46^\circ$ sin.		(ar. co.) 10.14307	(ar. co.) 10.14307
$b=72^\circ$ sin.	(ar. co.) 10.02179		(ar. co.) 10.02179
$c=68^\circ$ sin.	(ar. co.) 10.03283	(ar. co.) 10.03283	
$s=93^\circ$ sin.	9.99940	9.99940	9.99940
$s-a=47^\circ$ sin.	9.86413		
$s-b=21^\circ$ sin.		9.55433	
$s-c=25^\circ$ sin.			9.62595
	2)19.91815	2)19.72963	2)19.79021
cos.	9.95908	9.86482	9.89510

$$\tfrac{1}{2} A = 24^\circ\, 29', \ \tfrac{1}{2} B = 42^\circ\, 54', \ \tfrac{1}{2} C = 38^\circ\, 14'\, 24''.$$

Ans. $A = 48^\circ\, 58'$, $B = 85^\circ\, 48'$, $C = 76^\circ\, 28'\, 48''$.

II. By Table XXIII and equation (338),

$a - b = 26^\circ$	N. cos. 89879		
$c = 68^\circ$	N. cos. 37461		
	52418	log.	4.71948
$a = 46^\circ$		sin. (ar. co.)	0.14307
$b = 72^\circ$		sin. (ar. co.)	0.02179
$C = 5^h\, 5^m\, 55^s = 76^\circ\, 28'\, 45''$		log. Ris.	4.88434

2. Given in a spherical triangle the three sides equal to 3°, 4°, and 5°; to solve the triangle.

Ans. The three angles are $36^\circ\, 54'$, $53^\circ\, 10'$, and $90^\circ\, 2'$.

76. Napier obtained two theorems for the solution of a spherical triangle, when a side and the two adjacent angles are given, by which the two sides can be calculated without

the necessity of calculating the third angle. These theorems, which are given in § 79 and 80, can be obtained from equations (343 – 345) by the assistance of the following lemmas.

77. *Lemma.* If we have the equation

$$\frac{\text{tang. } M}{\text{tang. } N} = \frac{x}{y}, \tag{346}$$

we can deduce from it the following equation,

$$\frac{\sin. (M + N)}{\sin. (M - N)} = \frac{x + y}{x - y}. \tag{347}$$

Proof. We have from (7)

$$\text{tang. } M = \frac{\sin. M}{\cos. M}, \text{ and tang. } N = \frac{\sin. N}{\cos. N};$$

which, substituted in (346), give

$$\frac{\sin. M \cos. N}{\cos. M \sin. N} = \frac{x}{y}.$$

This equation is the same as the proportion

$$\sin. M \cos. N : \cos. M \sin. N = x : y;$$

hence, by the theory of proportions,

$$\sin. M \cos. N + \cos. M \sin. N : \sin. M \cos. N$$
$$- \cos. M \sin. N = x + y : x - y,$$

or, by (33) and (34),

$$\sin. (M + N) : \sin. (M - N) = x + y : x - y;$$

which may be written in the form of an equation, as in (347).

78. *Lemma.* If we have the equation

$$\text{tang. } M \text{ tang. } N = \frac{x}{y}; \tag{348}$$

we can deduce from it the equation

$$\frac{\cos. (M + N)}{\cos. (M - N)} = \frac{y - x}{y + x}. \tag{349}$$

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Proof. We have, by (348) and (7),

$$\frac{\sin. M \sin. N}{\cos. M \cos. N} = \frac{x}{y}.$$

This equation is the same as the proportion

$$\cos. M \cos. N : \sin. M \sin. N = y : x;$$

hence, by the theory of proportions,

$$\cos. M \cos. N - \sin. M \sin. N : \cos. M \cos. N + \sin. M \sin. N = y - x : x + y,$$

or, by (35) and (36),

$$\cos. (M + N) : \cos. (M - N) = y - x : y + x;$$

which may be written as in (349).

79. *Theorem. The sine of half the sum of two angles of a spherical triangle is to the sine of half their difference, as the tangent of half the side to which they are both adjacent is to the tangent of half the difference of the other two sides;* that is, in the spherical triangle ABC (figs. 32 and 33),

$$\sin. \tfrac{1}{2} (A + C) : \sin. \tfrac{1}{2} (A - C) = \text{tang}. \tfrac{1}{2} b : \text{tang}. \tfrac{1}{2} (a - c). \quad (350)$$

Proof. The quotient of (343) divided by (345) is, by an easy reduction,

$$\frac{\text{tang}. \frac{1}{2} A}{\text{tang}. \frac{1}{2} C} = \frac{\sin. (s - c)}{\sin. (s - a)}. \quad (351)$$

Hence, by § 77,

$$\frac{\sin. \frac{1}{2} (A + C)}{\sin. \frac{1}{2} (A - C)} = \frac{\sin. (s - c) + \sin. (s - a)}{\sin. (s - c) - \sin. (s - a)}. \quad (352)$$

If we make in equation (47)

$$A = s - c = \tfrac{1}{2} (a + b - c),$$
$$B = s - a = \tfrac{1}{2} (- a + b + c);$$

we have

$$A + B = b,$$
$$A - B = a - c;$$

and (47) becomes

$$\frac{\sin. (s - c) + \sin. (s - a)}{\sin. (s - c) - \sin. (s - a)} = \frac{\text{tang}. \frac{1}{2} b}{\text{tang}. \frac{1}{2} (a - c)}.$$

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This equation, substituted in the second number of (352), gives

$$\frac{\sin.\ \frac{1}{2}\,(A+C)}{\sin.\ \frac{1}{2}\,(A-C)} = \frac{\text{tang.}\ \frac{1}{2}\,b}{\text{tang.}\ \frac{1}{2}\,(a-c)}; \tag{353}$$

which is the same as (350).

80. *Theorem. The cosine of half the sum of two angles of a spherical triangle is to the cosine of half their difference, as the tangent of half the side to which they are both adjacent is to the tangent of half the sum of the other two sides;* that is, in the spherical triangle ABC (figs. 32 and 33),

$$\cos.\ \tfrac{1}{2}\,(A+C) : \cos\ \tfrac{1}{2}\,(A-C) = \text{tang.}\ \tfrac{1}{2}\,b : \text{tang.}\ \tfrac{1}{2}\,(a+c). \tag{354}$$

Proof. The product of (343) and (345) is, by a simple reduction,

$$\text{tang.}\ \tfrac{1}{2}\,A\ \text{tang.}\ \tfrac{1}{2}\,C = \frac{\sin.\ (s-b)}{\sin.\ s};$$

hence, by § 78,

$$\frac{\cos.\ \frac{1}{2}\,(A+C)}{\cos.\ \frac{1}{2}\,(A-C)} = \frac{\sin.\ s - \sin.\ (s-b)}{\sin.\ s + \sin.\ (s-b)}. \tag{355}$$

If in equation (47) inverted we make

$$A = s = \tfrac{1}{2}\,(a+b+c),$$

$$B = s - b = \tfrac{1}{2}\,(a-b+c);$$

we have

$$A + B = a + c,$$

$$A - B = b;$$

and (47) becomes

$$\frac{\sin.\ s - \sin.\ (s-b)}{\sin.\ s + \sin.\ (s-b)} = \frac{\text{tang.}\ \frac{1}{2}\,b}{\text{tang.}\ \frac{1}{2}\,(a+c)}.$$

This equation, substituted in (355), gives

$$\frac{\cos.\ \frac{1}{2}\,(A+C)}{\cos.\ \frac{1}{2}\,(A-C)} = \frac{\text{tang.}\ \frac{1}{2}\,b}{\text{tang.}\ \frac{1}{2}\,(a+c)}, \tag{356}$$

which is the same as (354).

81. *Scholium.* In using (350) and (354), the signs of the terms must be attended to by means of Pl. Trig. § 61.

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82. Examples.

1. Given in a spherical triangle two angles $= 158°$, and $= 98°$, and the included side $= 144°$; to find the other sides.

Solution. By (350),

$\frac{1}{2}(A+C) = 128°$	sin.	(ar. co.)	10.10347
$\frac{1}{2}(A-C) = 30°$	sin.		9.69897
$\frac{1}{2}\, b = 72°$	tang.		0.48822
$\frac{1}{2}(a-c) = 62\ 53'\ 1''$	tang.		0.29066

By (354),

$\frac{1}{2}(A+C) = 128°$	cos.	(ar. co.)	10.21066 n
$\frac{1}{2}(A-C) = 30°$	cos.		9.93753
$\frac{1}{2}\, b = 72°$	tang.		0.48822
$\frac{1}{2}(a+c) = 103°\ 0'\ 25''$	tang.		0.63641 n

Ans. $a = 165°\ 53'\ 26''$,
$c = 40°\ \ 7'\ 24''$.

2. Given in a spherical triangle two angles $= 170°$, and $= 2°$, and the included side $= 92°$; to find the other sides.

Ans. $a = 103°\ \ 6'\ 44''$,
$c = 11\ \ 17'\ 16''$.

83. *Problem. To solve a spherical triangle, when its three angles are given.*

Solution. If A, B, C are the angles of the given triangle, and a, b, c its sides, $180° - A$, $180° - B$, $180° - C$ are the sides of the polar triangle, and $180° - a$, $180° - b$, $180° - c$ the angles of the polar triangle, the sides are then given in the polar triangle; to find the angles. For this purpose we may use the formulas of the preceding problem.

84. *Corollary.* Applying (331) to the polar triangle gives

$$\cos. c = \frac{\cos. C + \cos. A \text{ and } B}{\sin. A \sin. B}. \tag{357}$$

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85. *Corollary*. Equations (335-337) give, for the polar triangle, if we put

$$S = \tfrac{1}{2}(A + B + C), \tag{358}$$

and use (98 and 99),

$$\sin. \tfrac{1}{2} a = \surd \left(\frac{-\cos. S \cos. (S - A)}{\sin. B \sin. C} \right), \tag{359}$$

$$\sin. \tfrac{1}{2} b = \surd \left(\frac{-\cos. S \cos. (S - B)}{\sin. A \sin. C} \right), \tag{360}$$

$$\sin. \tfrac{1}{2} c = \surd \left(\frac{-\cos. S \cos. (S - C)}{\sin. A \sin. B} \right). \tag{361}$$

86. *Corollary*. Equations (340-342), applied to the polar triangle, give

$$\cos. \tfrac{1}{2} a = \surd \left(\frac{\cos. (S-B) \cos. (S-C)}{\sin. B \sin. C} \right), \tag{362}$$

$$\cos. \tfrac{1}{2} b = \surd \left(\frac{\cos. (S-A) \cos. (S-C)}{\sin. A \sin. C} \right). \tag{363}$$

$$\cos. \tfrac{1}{2} c = \surd \left(\frac{\cos. (S-A) \cos. (S-B)}{\sin. A \sin. B} \right). \tag{364}$$

87. *Corollary*. Equations (343-345), applied to the polar triangle, give

$$\text{tang.} \tfrac{1}{2} a = \surd \left(\frac{-\cos. S \cos. (S-A)}{\cos. (S-B) \cos. (S-C)} \right), \tag{365}$$

$$\text{tang.} \tfrac{1}{2} b = \surd \left(\frac{-\cos. S \cos. (S-B)}{\cos. (S-A) \cos. (S-C)} \right), \tag{366}$$

$$\text{tang.} \tfrac{1}{2} c = \surd \left(\frac{-\cos. S \cos. (S-C)}{\cos. (S-A) \cos. (S-B)} \right). \tag{367}$$

88. *Corollary*. Equation (332), applied to the polar triangle, is

$$2 (\sin. \tfrac{1}{2} c)^2 = \frac{-\cos. C - \cos. (A + B)}{\sin. A \sin. B}, \tag{368}$$

which may be used like equation (338).

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89. Example.

Given in the spherical triangle ABC, the three angles equal to $89°$, $5°$, and $88°$; to solve the triangle.

Ans. The three sides are $53° \ 10'$, $4°$, and $53° \ 8'$.

90. *Theorem. The sine of half the sum of two sides of a spherical triangle is to the sine of half their difference, as the cotangent of half the included angle is to the tangent of half the difference of the other two angles,* that is, in ABC (figs. 32 and 33),

$$\sin. \tfrac{1}{2}(a+c) : \sin. \tfrac{1}{2}(a-c) = \text{cotan.} \tfrac{1}{2} B : \text{tang.} \tfrac{1}{2}(A-C). \quad (369)$$

Proof. This theorem is at once obtained by applying § 79 to the polar triangle.

91. *Theorem. The cosine of half the sum of two sides of a triangle is to the cosine of half their difference, as the cotangent of half the included angle is to the tangent of half the sum of the other two angles,* or in (figs. 32 and 33),

$$\cos. \tfrac{1}{2}(a+c) : \cos. \tfrac{1}{2}(a-c) = \text{cotan.} \tfrac{1}{2} B : \text{tang.} \tfrac{1}{2}(A+C). \quad (370)$$

Proof. This theorem is at once obtained by applying § 80 to the polar triangle.

92. *Corollary.* These two theorems, similar to § 79 and § 80, were given by Napier for the solution of the case, in which two sides and the included angle are given. By means of them the other two angles can be found without the necessity of calculating the third side. In using them, regard must be had to the signs of the terms by means of Pl. Trig. § 61.

93. Examples.

1. Given in a spherical triangle two sides $= 149°$, and $= 49°$, and the included angle $= 88°$; to find the other angles.

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Solution. By § 90,

$\frac{1}{2}(a+c) = 99°$	sin.	(ar. co.)	10.00538
$\frac{1}{2}(a-c) = 50°$	sin.		9.88425
$\frac{1}{2}B \quad = 44°$	cotan.		0.01516
$\frac{1}{2}(A-C) = 38° \; 46' \; 10''$	tang.		9.90479

By § 91,

$\frac{1}{2}(a+c) = 99°$	cos.	(ar. co.)	10.80567 *n*
$\frac{1}{2}(a-c) = 50°$	cos.		9.80807
$\frac{1}{2}B \quad = 44°$	cotan.		0.01516
$\frac{1}{2}(A+C) = 103° \; 13' \; 31''$	tang.		0.62890 *n*

Ans. $A = 141° \; 59' \; 41''$,

$C = 64° \; 26' \; 21''$.

2. Given in a spherical triangle two sides $= 13°$, and $= 9°$, and the included angle $= 176$; to find the other angles.

Ans. $2° \; 13' \; 12''$, and $1° \; 51' \; 14''$.

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SPHERICAL ASTRONOMY.

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SPHERICAL ASTRONOMY.

CHAPTER I.

THE CELESTIAL SPHERE AND ITS CIRCLES.

1. *Astronomy* is the science which treats of the heavenly bodies.

2. *Mathematical Astronomy* is the science which treats of the positions and motions of the heavenly bodies.

The elements of position of a heavenly body are (Geo. § 8) distance and direction.

3. *Spherical Astronomy* regards only one of the elements of position, namely, direction, and usually refers all directions to the centre of the earth.

4. In spherical astronomy, all the stars may, then, be regarded as at the same distance from the earth's centre, upon the surface of a sphere, which is called the *celestial sphere.*

Upon this imaginary sphere are supposed to be drawn various circles, which are divided into the well known classes of *great* and *small* circles. [B. p. 47.]

"All angular distances on the surface of the sphere, to an eye at the centre, are measured by arcs of *great* circles." [B. p. 48.]

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5. "*Secondaries* to a great circle are great circles which pass through its poles, and are consequently perpendicular to it." [B. p. 48.]

6. "If the plane of the *terrestrial equator* be produced to the celestial sphere, it marks out a circle called the *celestial equator;* and if the axis of the earth be produced in like manner, it becomes the *axis* of the celestial sphere; and the points of the heavens, to which it is produced, are called the *poles*, being the poles of the celestial equator."

"The star nearest to the north pole is called the *north pole star.*" [B. p. 48.]

7. "Secondaries to the celestial equator are called *circles of declination;* of these, 24, which divide the equator into equal parts of 15° each, are called *hour circles.*"

"Small circles, parallel to the celestial equator, are called *parallels* of declination." [B. p. 48.]

The parallels of declination correspond, therefore, to the terrestrial parallels of latitude, and the circles of declination to the terrestrial meridians. A certain point of the celestial equator has been fixed by astronomers, and is called the *vernal equinox.* The circle of declination, which passes through the vernal equinox, bears the same relation to other circles of declination, which the first meridian does to other terrestrial meridians.

8. "The *declination* of a star is its angular distance from the celestial equator," measured upon its circle of declination. [B. p. 49.]

9. The *right ascension* of a star is the arc of the equator intercepted between its circle of declination and the vernal equinox. [B. p. 49.]

Right ascension is either estimated in degrees, minutes, &c. from 0° to 360°; or in hours, minutes, &c. of time, 15 degrees being allowed for each hour, as in Sph. Trig. § 3.

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The positions of the stars are completely determined upon the celestial sphere, when their right ascensions and declinations are known. Catalogues of the stars have accordingly been given, containing their right ascensions and declinations. [B. Table VIII. p. 80.]

10. "The *sensible horizon* is that circle in the heavens, whose plane touches the earth at the spectator."

"The *rational horizon* is a great circle of the celestial sphere parallel to the sensible horizon." [B. p. 48.]

11. The radius, which is drawn to the observer, is called the *vertical* line.

The point, where the vertical line meets the celestial sphere *above* the observer, is called the *zenith;* the opposite point, where this line meets the sphere *below* the observer, is called the *nadir.*

Hence the vertical line is a radius of the celestial sphere perpendicular to the horizon; and the zenith and nadir are the poles of the horizon. [B. p. 48.]

12. Circles whose planes pass through the vertical line are called *vertical* circles. [B. p. 48.]

The vertical circles are secondaries to the horizon.

13. The vertical circle at any place, which is also a circle of declination, is called the *celestial meridian* of that place. [B. p. 48.]

The plane of the celestial meridian of a place is the same with that of the terrestrial meridian.

14. The points, where the celestial meridian cuts the horizon, are called the *north* and *south* points. [B. p. 48.]

The north point corresponds to the north pole, and the south point to the south pole.

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15. The vertical circle, which is perpendicular to the meridian, is called the *prime vertical.* [B. p. 48.]

16. The points, where the prime vertical cuts the horizon, are called the *east* and *west* points. [B. p. 48.]

"To an observer, whose face is directed towards the south, the east point is to his left hand, and the west to his right hand. Hence the east and west points are 90° distant from the north and south. These four are called the *cardinal* points."

"The meridian of any place divides the heavens into two hemispheres, lying to the east and west; that lying to the east is called the *eastern* hemisphere, and the other the *western* hemisphere."

17. The *altitude* of a star is its angular distance from the horizon, measured upon the vertical circle passing through the star. [B. p. 48.]

18. The *azimuth* of a star is the arc of the horizon intercepted between its vertical circle and the north or south point. [B. p. 48.]

A star may be found without difficulty, when its altitude and azimuth are known. But these elements of position are constantly varying.

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CHAPTER II.

THE DIURNAL MOTION.

19. "STARS are distinguished into two kinds, *fixed* and *wandering*." [B. p. 45.]

Most of the stars are fixed, that is, retain constantly almost the same relative position; so that the same celestial globes and maps continue to be accurate representations of the firmament for many years. This is a fact of fundamental importance, and furnishes the fixed points for arriving at a complete knowledge of the celestial motions. Small changes of position have, indeed, been detected even in the fixed stars, as will be shown in the course of this treatise; but these changes are too small to disturb the general fact; they are, indeed, too small ever to have been detected, if the positions of the stars had been subject to great variations.

20. Of the wandering stars there are eleven, which are called *planets*. They are *Mercury* (☿), *Venus* (♀), *the Earth* (⊕), *Mars* (♂), *Vesta* (⚶), *Juno* (⚵), *Pallas* (⚴), *Ceres* (⚳), *Jupiter* (♃), *Saturn* (♄), and *Uranus* (♅). [B. p. 45.]

21. For the sake of remembering the stars with greater ease, they have been divided into groups called *constellations;* and to give distinctness to the constellations, they have been supposed to be circumscribed by the outlines of some figure which they were imagined to resemble. [B. p. 45.]

The stars have also been distinguished according to their brilliancy, as of the *first*, *second*, &c. magnitude.

Proper names have been given to the constellations and to the most remarkable stars.

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The catalogues and the maps of the stars are now so accurate, that no new star could appear without being detected; and any change in the place of any of the larger stars would be immediately discovered.

22. All the stars appear to have a common motion, by which they are carried round the earth from east to west in 24 hours. This rotation of the heavens, or of the celestial sphere, is called the *diurnal motion.*

By its diurnal motion, the celestial sphere rotates, with the most perfect uniformity, about its axis. The pole star would, therefore, if it were exactly at the pole, remain stationary; but since it is not exactly at the pole, it revolves in a very small parallel of declination about the stationary pole.

Any star in the equator revolves in the plane of the equator, and all other stars revolve in the planes of the parallels of declination in which they are situated.

If O (fig. 34) is the place of the observer, $NESW$ his horizon, Z his zenith, P and P' the poles, the star which is at the distance from P,

$$PM = PM'$$

will appear to describe the circumference $MH'M'H$. It will rise in the east at H and set at H', if the distance PM' from the pole is greater than the altitude PN of the pole. But if its distance from the pole

$$PL = PL'$$

is less than PN, the star will not set, but will describe a circle above the horizon; and if its distance from the pole

$$PG = PG'$$

is greater than the greatest distance PS from the pole to the horizon, the star will never rise so as to be seen by the observer at O, but will describe a circle below the horizon.

23. The time which it takes a star to pass from any position round again to the same position, is called a *sideral* day, that is, literally, a star-day. This day is divided into 24 hours,

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and clocks regulated to this time are said to denote *sideral time.* [B. p. 147.]

24. Each point of the celestial equator passes the meridian once in a sideral day; and the arc contained between two hour circles passes it in a sideral hour. The sideral time, therefore, which has elapsed since the vernal equinox was upon the equator, is equal to the right ascension of the meridian expressed in time. [B. p. 208.]

The meridian changes its right ascension at each instant, precisely *as if* the celestial sphere were stationary, while the observer, with his meridian and zenith, is carried uniformly round the earth's centre from west to east once in a sideral day.

25. The angle ZPB (fig. 35), which the circle of declination of the star makes with the meridian, is called its *hour angle.*

While the star moves from the point of C in the meridian to the point B with an uniform motion, the arc CP is carried to the position PB, and the angle CPB is described with an uniform motion. This angle converted into time is, then, the sideral time since the passage of the star over the meridian.

26. *Corollary.* The difference of the right ascensions of the star and of the meridian is the hour angle of the star.

27. The distance of a star from the east or west points of the horizon, at the time of its rising or setting, is the *amplitude* of the star. [B. p. 48.]

28. *Problem. To find the altitude and azimuth of a star, when its declination and hour angle are known, and also the latitude of the place.*

Solution. If P (fig. 35) is the pole, Z the zenith, and B the star, we have

$$PZ = \text{polar dist. of zenith} = \text{co. latitude} = 90^\circ - L,$$

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$PN = 90° - PZ = L,$

$PB =$ polar dist. of star $= p,$

$=$ co. declination of star, when it is on the same side of the equator with the pole.

$= 90° +$ declination of star, when it is on the different side of the equator from the pole.

$= 90° \mp D,$

$ZB =$ zenith dist. of star $= z,$

$=$ co. altitude of star, when it is above the horizon.

$= 90° +$ depression of star, when it is below the horizon.

$ZPB = *$'s hour angle $= h,$

$PZB =$ azimuth of star counted from the direction of the elevated pole.

$= a =$ azimuth, when less than $90°$,

$= 180° -$ azimuth, when greater than $90°$.

There are, then, given in the spherical triangle PZB, the two sides PZ and PB, and the included angle ZPB; so that the side BZ and the angle PZB can be calculated by Sph. Trig. § 45.

If we let fall the perpendicular BC upon PZ,

$$\text{tang. } PC = \cos. h \text{ tang. } (90° \mp D) = \pm \cos. h \text{ cotan. } D \qquad (371)$$

$$CZ = PZ - PC = 90° - (L + PC),$$

$$\text{or} \qquad = PC - PZ = (L + PC) - 90°. \qquad (372)$$

Hence, by (300),

$$\cos. PC : \sin. (L + PC) = \pm \sin. D : \cos. z; \qquad (373)$$

in which formulas the upper sign is used when the star is upon the same side of the equator with the elevated pole, that is, when D and L are of the same name; and, by (301),

$$\sin. PC : \pm \cos. (L + PC) = \text{cotan. } h : \text{cotan. } a. \qquad (374)$$

29. *Corollary.* When the altitude and azimuth are both to be found, the calculation by the above method is as short as by any

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other; but when, as is usually the case, the altitude only is required, the following method is preferable.

We have

$$PZ + PB = 180° - L \mp D = 180° - (L \pm D)$$

$$PB - PZ = \mp D + L = (L \mp D);$$

whence, by (308) and (309),

$$\cos. z = - \cos. (L \pm D) + 2 \cos. D \cos. L (\cos. \tfrac{1}{2} h)^2 \quad (375)$$

$$\cos. z = \cos. (L \mp D) - 2 \cos. D \cos. L (\sin. \tfrac{1}{2} h)^2, \quad (376)$$

which may be used at once, and (376) may be calculated by the aid of the column of Rising in Table XXIII. The rule obtained from (376) is the same with that on p. 250 of the Navigator, remembering that when the star is above the horizon

$$\cos. z = \sin. \ast\text{'s alt.} \quad (377)$$

But when the star is below the horizon

$$\cos. z = - \sin. \ast\text{'s depression.} \quad (378)$$

30. *Corollary.* If the given hour angle is $6^h = 90°$, the problem is at once reduced to the solution of a right triangle. We in this case have, by Napier's Rules,

$$\cos. z = \sin. L \cos. p,$$

or

$$\sin. \ast\text{'s alt.} = \pm \sin. L \sin. D \quad (379)$$

$$\text{cotan. } a = \cos. L \text{ cotan. } p$$

$$\text{cotan. } \ast\text{'s azimuth} = \pm \cos. L \text{ tang. } D. \quad (380)$$

The upper sign is to be used in formulas (379) and (380), when the declination is of the same name with the latitude; otherwise the lower sign. In the former case, therefore, the star is above the horizon when its hour angle is six hours, and on the same side of the prime vertical with the elevated pole; but, in the latter case, it is below the horizon, and on the same side of the prime vertical with the depressed pole.

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31. *Corollary.* If the star is in the celestial equator, as in (fig. 36), we have in the right triangle BZQ,

$$BQ = BPQ = h$$

$$ZQ = L$$

$$QZB = 180° - a$$

whence
$$\cos. z = \cos. L \cos. h,$$

or
$$\sin. \text{✱'s alt.} = \cos. L \cos. h \tag{381}$$

$$\text{cotan.}\,(180° - a) = \sin. L\ \text{cotan.}\ h,$$

or
$$\text{cotan.}\ a = - \sin. L\ \text{cotan.}\ h. \tag{382}$$

Hence, if the hour angle is less than six hours, the star which moves in the celestial equator is above the horizon, and on the same side of the prime vertical with the depressed pole; but if the hour angle is greater than six hours, this star is below the horizon, and on the same side of the prime vertical with the elevated pole.

32. *Corollary.* If the place is at the equator, as in (fig. 37), the celestial equator of ZE is the prime vertical, so that if the hour circle PB is produced to C, we have in the right triangle ZBC,

$$ZC = ZPB = h$$

$$BZC = 90° - a$$

$$BC = D,$$

whence
$$\cos. z = \cos. D \cos. h,$$

or
$$\sin. \text{✱'s alt.} = \cos. D \cos. h \tag{383}$$

$$\text{cotan.}\,(90° - a) = \sin. h\ \text{cotan.}\ D,$$

or
$$\text{tang.}\ a = \sin. h\ \text{cotan.}\ D; \tag{384}$$

so that the star is above the horizon when the hour angle is less than six hours, and below the horizon when the hour angle is greater than six hours.

33. Examples.

1. Find the altitude and azimuth of Aldebaran to an observer at Boston, in the year 1830, when the hour angle of this star is $3^h\ 25^m\ 12^s$.

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Solution. We find by Tables VIII and LIV

$D = 16° \; 10' \text{ N.} \qquad L = 42° \; 21' \text{ N.}$

Hence

$h = 3^h \; 25^m \; 12^s$	log. col. Ris.			4.57375
$L = 42° \; 21'$	cos.			9.86867
$D = 16° \; 10'$	cos.			9.98248
		26601		4.42490
$L - D = 26° \; 10'$	nat. cos.	89752		
alt. $= 39° \; 10'$	nat. sin.	63151	sec.	10.11052
	$h = 51° \; 18'$		sin.	9.89233
		D	cos.	9.98248
azimuth from South $= 75° \; 11'$			sin.	9.98533

2. Find the altitude and azimuth of Aldebaran at Boston, in the year 1830, six hours after it has passed the meridian.

Solution. By formulas (379) and (380),

$L = 42° \; 21'$	sin.	9.82844	cos.	9.86867
$D = 16° \; 10'$	sin.	9.44472	tang.	9.46224
alt. $= 10° \; 49'$	sin.	9.27316		
azimuth from north $= 77° \; 54'$			cotan.	9.33091

3. Find the altitude and azimuth of a star in the celestial equator to an observer at Boston, when the hour angle of the star is $3^h \; 25^m \; 12^s$.

Solution. By formulas (381) and (382),

$L = 42° \; 21'$	cos.	9.86867	sin.	9.82844
$h = 51° \; 18'$	cos.	9.79605	cotan.	9.90371
alt. $= 27° \; 31'$	sin.	9.66472		
azimuth from South $= 61° \; 39'$			cotan.	9.73215

4. Find the altitude and azimuth of Aldebaran to an observer at the equator, in the year 1830, when the hour angle of the star is $3^h \; 25^m \; 12^s$.

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Solution. By formulas (383) and (384),

$D = 16^\circ\ 10'$	cos.	9.98248	cotan.	10.53776
$h = 51^\circ\ 18'$	cos.	9.79605	sin.	9.89233
alt. $= 36^\circ\ 54'$	sin.	9.77853		
azimuth from North $= 69^\circ\ 37'$			tang.	10.43009

5. Find the altitude and azimuth of Fomalhaut to an observer at Boston, in the year 1840, when its hour angle is $2^h\ 3^m\ 20^s$.

Ans. Its altitude . . . $= 11^\circ\ 59'$.
Its azimuth from the South $= 26^\circ\ 51'$.

6. Find the altitude and azimuth of Dubhe to an observer at Boston, in the year 1840, when its hour angle is $9^h\ 30^m$.

Ans. Its altitude . . . $= 19^\circ\ 11'$.
Its azimuth from the North $= 17^\circ\ 15'$.

7. Find the altitude and azimuth of Fomalhaut to an observer at Boston, in the year 1840, when its hour angle is 6^h.

Ans. Its depression below the horizon $= 19^\circ\ 58'$.
Its azimuth from the South $= 66^\circ\ 30'$.

8. Find the altitude and azimuth of Dubhe to an observer at Boston, in the year 1840, when its hour angle is 6^h.

Ans. Its altitude . . . $= 36^\circ\ 44'$.
Its azimuth from the North $= 35^\circ\ 2'$.

9. Find the altitude and azimuth of a star in the celestial equator to an observer at Stockholm, when its hour angle is $2^h\ 3^m\ 20^s$.

Ans. Its altitude . . . $= 25^\circ\ 58'$.
Its azimuth from the South $= 34^\circ\ 45'$.

10. Find the altitude and azimuth of a star in the celestial equator to an observer at Stockholm, when the hour angle is $9^h\ 30^m$.

Ans. Its depression below the horizon $= 23^\circ\ 51'$.
Its azimuth from the North $= 41^\circ\ 45'$.

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11. Find the altitude and azimuth of Fomalhaut to an observer at the equator, in the year 1840, when its hour angle is $2^h\ 3^m\ 20^s$.

Ans. Its altitude . . . $= 47° 45'$.
Its azimuth from the South $= 41° 4'$.

12. Find the altitude and azimuth of Dubhe to an observer at the equator, in the year 1840, when its hour angle is $9^h\ 30^m$.

Ans. Its depression below the horizon $= 21° 24'$.
Its azimuth from the North $= 17° 30'$.

34. In the triangle ZPB (fig. 35) other parts might be given instead of the two sides ZP, PB, and the included angle P, and the triangle might be resolved. Of the problems thus derived, we shall only, for the present, consider two cases.

35. *Problem. To find a given star's hour angle and altitude, when it is upon the prime vertical.*

Solution. The angle PZB is, in this case, a right angle, and if we use the preceding notation, we have

$$\cos. h = \text{cotan. } L \text{ cotan. } p = \pm \text{ cotan. } L \text{ tang. } D \qquad (385)$$

$$\cos. z = \cos. p \text{ cosec. } L,$$

or

$$\sin. \ast\text{'s alt.} = \pm \sin. D \text{ cosec. } L; \qquad (386)$$

so that when the declination and latitude are of the same name, the hour angle is less than 6 hours, and the star is above the horizon; but when the declination and latitude are of different names, the hour angle is greater than 6 hours, and the star is below the horizon.

36. *Scholium.* The problem is, by Sph. Trig. § 27, impossible, when the declination is greater than the latitude; so that, in this case, the star is never exactly east or west of the observer.

37. *Scholium.* The problem is, by Sph. Trig. § 28, indeterminate, when the latitude and declination are both equal to zero; so that, in this case, the star is always upon the prime vertical.

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38. Examples.

1. Find the hour angle and altitude of Aldebaran, when it is exactly east or west of an observer at Boston, in the year 1840.

Ans. The hour angle $= 4^h\ 45^m\ 44^s$.
The altitude $= 24°\ 26'$.

2. Find the hour angle and altitude of Fomalhaut, when it is exactly east or west of an observer at Boston, in the year 1840.

Ans. The hour angle . . $= 8^h\ 40^m\ 51^s$.
The depression below the horizon $= 48°\ 49'$.

3. Find the hour angle and altitude of Dubhe, when it is exactly east or west of an observer at Boston, in the year 1840.

Ans. Dubhe is never upon the prime vertical of Boston.

4. Find the hour angle and altitude of Canopus, when it is exactly east or west of an observer at Boston, in the year 1840.

Ans. Canopus is never upon the prime vertical of Boston.

39. *Problem. To find the hour angle and amplitude of a star, when it is in the horizon.*

Solution. In this case the side ZB (fig. 35) of the triangle ZPB is 90°. The corresponding angle of the polar triangle is, therefore, a right angle, and the polar triangle is a right triangle, of which the other two angles are

$$180° - PZ = 180° - (90° - L) = 90° + L,$$

and

$$180° - PB = 180° - (90° \mp D) = 90° \pm D.$$

The hypothenuse of the polar triangle is $180° - h$, and the leg, opposite the angle, $90° \pm D$, is $180° - a$.

Hence, by Sph. Trig. § 40, and Pl. Trig. § 60 and 62,

$$-\cos. h = \pm \text{tang. } L \text{ tang. } D,$$

or

$$\cos. h = \mp \text{tang. } L \text{ tang. } D \qquad (387)$$

$$-\cos. a = \mp \sin. D \sec. L,$$

or

$$\cos. a = \pm \sin. D \sec. L; \qquad (388)$$

in which the upper sign is used when the latitude and declination

have the same name, and the lower sign when they have different names; so that in the former case the hour angle is greater than 6 hours, and the azimuth is counted from the direction of the elevated pole; but in the latter case, the hour angle is less than 6 hours, and the azimuth is counted from the direction of the depressed pole. The amplitude is the difference between the azimuth a and $90°$. Hence

$$\cos. \ast\text{'s azim.} = \sin. \ast\text{'s amp.} = \sin. D \sec. L. \tag{389}$$

40. *Scholium.* The problem is, by Sph. Trig. § 41, impossible, when the sum of the declination and latitude is greater than $90°$; so that, in this case, the star does not rise or set.

41. Examples.

1. Find the hour angle and amplitude of Aldebaran, when it rises or sets, to an observer at Boston, in the year 1840.

Ans. The hour angle $= 7^h\ 1^m\ 21^s$.
The amplitude $= 22°\ 9'$ N.

2. Find the hour angle and amplitude of Fomalhaut, when it rises or sets, to an observer at Boston, in the year 1840.

Ans. The hour angle $= 3^h\ 50^m\ 18^s$.
The amplitude $= 43°\ 19'$ S.

3. Find the hour angle and amplitude of Dubhe, when it rises or sets, to an observer at Boston, in the year 1840.

Ans. Dubhe neither rises nor sets at Boston.

4. Find the hour angle and amplitude of Canopus, when it rises or sets, to an observer at Boston, in the year 1840.

Ans. Canopus neither rises nor sets at Boston.

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CHAPTER III.

THE MERIDIAN.

42. The intersection of the plane of the meridian with that of the horizon, is called the *meridian line.*

43. *Problem. To determine the meridian line.*

Solution. First Method. Stars obviously rise to their greatest altitude in the plane of the meridian; so that if their progress could be traced with perfect accuracy, and the instant of their rising to their greatest height be observed, the direction of the meridian line could be exactly determined. But stars, when they are at their greatest height, change their altitude so slowly, that this method is of but little practical value.

Second Method. A star is evidently at equal altitudes when it is at equal distances from the meridian on opposite sides of it. If, therefore, the direction and altitude of a star are observed before it comes to the meridian; and if its direction is also observed, when it has descended again to the same altitude, after passing the meridian; the horizontal line, which bisects the angle of the two horizontal lines drawn in the direction thus determined, is the meridian line.

Third Method. [B. p. 147.] The time which elapses between the superior and inferior passage of a star over the meridian is just half of a sideral day. If, then, a telescope were placed so as to revolve on a horizontal axis in the plane of the meridian, the two intervals of time between three successive passages of a star over the central wire, must be exactly equal. But if the vertical plane of the telescope is not that of the meridian, these two intervals will not be equal, and the position of the telescope must be changed until they become equal.

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Thus, if $Z\,M\,m\,N$ (fig. 38) is the plane of the meridian, $Z\,S\,s\,T$ that of the vertical circle described by the telescope, $M\,S\,W\,s\,m\,E$ the circle of declination described by the star about the pole P; this star will be observed at the points S and s instead of at the points M and m. Now the star describes the circle of declination with an uniform motion, and therefore the arc SP moves uniformly with the star around the pole, so that the angle SPM is proportional to the time of its description; that is, the angle SPM, reduced to time, denotes the sideral time of its description.

Let then

$T =$ the sideral time of describing the arc SM,

$t =$ the sideral time of describing the arc $s\,m$,

$I =$ interval from the observation at S to that at s,

$i =$ interval from the observation at s to that at S,

$\delta i =$ the difference of these two intervals;

we have then, in sideral time,

$$I = 12^h - T - t = 12^h - (T + t)$$
$$i = 12^h + T + t = 12^h + (T + t)$$
$$\delta i = i - I = 2\,(T + t); \qquad (390)$$

so that if T and t were equal to each other, and they are nearly so in the case of the pole-star, we should have

$$\delta i = 4\,T = 4\,t$$
$$T = t = \tfrac{1}{4}\,\delta i;$$

that is, *the time of describing the arc MS or m s is nearly one quarter part of the difference between the intervals.*

But the error of this result can be calculated without much difficulty. For this purpose, let

$L =$ the latitude of the place $= 90° - PZ$,

$p =$ the polar distance of the star $= PS = P\,s$,

$a =$ the azimuth of $ZST = TN = TZN$.

The arcs MS and $m\,s$ are so small, that they do not differ sensibly from the arcs of great circles drawn from S and s perpendicular to ZPN.

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If, then, in the two right triangles PSM and ZSM, PM and ZM are the middle parts, SM, co. SZM, and co. SPM are the adjacent parts, so that

$$\begin{aligned} \sin. PM : \sin. ZM &= \text{cotan. } SPM : \text{cotan. } SZM \\ &= \frac{1}{\text{tang. } SPM} : \frac{1}{\text{tang. } SZM} \\ &= \text{tang. } SZM : \text{tang. } SPM. \end{aligned}$$

But
$$ZM = ZP - PM = 90^\circ - L - p,$$

and the angles SZM and SPM are so small, that they are sensibly proportional to their tangents, whence

$$\sin. p : \cos. (p + L) = a : SPM, \tag{391}$$

or
$$\begin{aligned} a : SPM &= \sin. p : \cos. p \cos. L - \sin. p \sin. L \\ &= 1 : \text{cotan. } p \cos. L - \sin. L, \end{aligned}$$

and if T is expressed in sideral hours

$$T \,.\, 15^\circ = SPM = a \text{ cotan. } p \cos. L - a \sin. L.$$

In like manner, we find

$$t \,.\, 15^\circ = s\,P\,m = a \text{ cotan. } p \cos. L + a \sin. L.$$

Hence, by (390),

$$(T + t)\, 15^\circ = \tfrac{1}{2}\, \delta\, i \,.\, 15^\circ = 2\, a \text{ cotan. } p \cos. L$$

$$a \text{ cotan. } p \cos. L = \tfrac{1}{4}\, \delta\, i \,.\, 15^\circ$$

$$T \,.\, 15^\circ = \tfrac{1}{4}\, \delta\, i \,.\, 15^\circ - a \sin. L$$

$$t \,.\, 15^\circ = \tfrac{1}{4}\, \delta\, i \,.\, 15^\circ + a \sin. L$$

$$a = \tfrac{1}{4}\, \delta\, i \,.\, 15^\circ \text{ tang. } p \sec. L \tag{392}$$

$$T = \tfrac{1}{4}\, \delta\, i - \tfrac{1}{4}\, \delta\, i \text{ tang. } p \text{ tang. } L$$

$$t = \tfrac{1}{4}\, \delta\, i + \tfrac{1}{4}\, \delta\, i \text{ tang. } p \text{ tang. } L,$$

so that the correction is

$$\tfrac{1}{4}\, \delta\, i \text{ tang. } p \text{ tang. } L, \tag{393}$$

which is to be added to the quarter interval at the lower transit; and to be subtracted from the quarter interval at the upper transit.

This correction is proportional to the quarter interval, so that if it is computed for any supposed value of this interval, it may be com-

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puted for any other interval by a simple proportion. Now Table A, page 151, of the Navigator, is the value of this correction, whenthe interval is 1000^{s}. It may be observed, that it is not necessary that this time should be sideral time, because all the terms of the values of T and t are expressed in the same time, which may be that of the clock.

The azimuth a is given in Table B [B. p. 151], and may be computed from the formula (392). But the interval in the formula is supposed to be sideral time, whereas the time of the table is that called *solar time*, to which clocks are usually regulated, and which is soon to be described; all that need be known for the present is, that an interval of sideral time is reduced to solar time by Table LII of the Navigator, or by the formula

$$\frac{\text{an interval of solar time}}{\text{an interval of sideral time}} = 0.9972695. \qquad (394)$$

Fourth Method. [B. p. 149.] This method of determining the meridian is by means of two known circumpolar stars, which differ nearly 12 hours in right ascension. The upper passage of one of these stars is to be observed, and the lower passage of the other. Then any deviation in the plane of the instrument from the meridian, will evidently produce contrary effects upon the observed times of transit, exactly as in the upper and lower transits of the same star. The time, which elapses between the two observations, will differ from the time which should elapse by the sum of the effects of the deviation upon the two stars. In the use of this method, therefore, the time of the clock must be known, so that it can readily be reduced to sideral time.

The deviations in the time of passage of a star, corresponding to any azimuth, can be calculated by means of equation (391). For this formula gives for the time of describing the arc SM

$$T \,.\, 15^{\circ} = a \cos.\ (p + L) \text{ cosec. } p,$$

or

$$T = \tfrac{1}{15}\ a \cos.\ (p + L) \text{ cosec. } p; \qquad (395)$$

which may be used if T is expressed in sideral seconds, and the arc a in seconds of space. But if T is expressed in solar time, we have, by (394),

$$T = 0.0664846\ a \cos.\ (p + L) \text{ cosec. } p. \qquad (396)$$

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In the same way the value of t for an inferior passage is found to be

$$t = 0.0664846\, a \cos. (p - L) \operatorname{cosec.} p. \qquad (397)$$

Now, since these values of T and t are proportional to the azimuth a, their values may be computed for a given value of the azimuth, as $1000''$, and arranged in a table like Table C, p. 152 of the Navigator, and their values for any other azimuth can be obtained by a simple proportion.

Fifth Method. [B. p. 149.] This method consists in observing the transits of two stars, which differ but little in right ascension. The error in the position of the telescope is, in this case, equal to the difference in the errors of the observed transits, instead of the sum, as in the preceding method.

44. In making calculations where angles are introduced as factors, some labor, in reducing them to the same denomination, is often saved by means of a table of Proportional Logarithms, such as Table XXII of the Navigator.

This table was particularly designed for reducing lunar distances, given in the Nautical Almanac, for every 3 hours to any intermediate time. It contains, on this account, the logarithm of the ratio of 3 hours to each angle expressed in time; that is, if A is the angle.

$$\text{Prop. log. } A = \log. \frac{3^h}{A} = \log. 3^h - \log. A = \log. 180^m - \log. A$$
$$= \log. 10800^s - \log. A, \qquad (398)$$

so that if A in the second member is reduced to seconds,

$$\text{Prop. log. } A = 4.03342 - \log. A \text{ in seconds}; \qquad (399)$$

neglecting the right hand figure, so as to retain only four decimal places. This agrees with the explanation of the table in the Introduction to the Navigator; and it is evident that it is immaterial whether the angles, whose ratios are sought, are given in time or in degrees, &c.

Suppose, now, that the logarithm of the ratio of two angles is sought, A and a; we have, evidently,

$$\log. \frac{A}{a} = \log. A - \log. a = \text{Prop. log. } a - \text{Prop. log. } A; \qquad (400)$$

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so that if this ratio, which we will denote by M, were known, and if a were known, A might be calculated by the formula

$$\text{Prop. log. } A = \text{Prop. log. } a - \log. M$$
$$= \text{Prop. log. } a + (\text{ar. co.}) \log. M; \qquad (401)$$

which is, therefore, the formula for calculating the value of A, given by the equation

$$A = a\,M. \qquad (402)$$

Finally, the use of formula (401) is facilitated by remembering that *the arithmetical complements of the logarithms of the sine, cosine, tangent, cotangent, secant, and cosecant of an angle, are respectively the logarithms of its cosecant, secant, cotangent, tangent, cosine, and sine.*

45. Examples.

1. Calculate the proportional logarithm of 0° 5′ 45″.

Solution. By (399),		4.03342
0° 5′ 45″ = 345″.		2.53782
Prop. log. 5′ 45″	=	1.4956

as in Table XXII.

2. Calculate the corrections of Tables A and B [B. p. 151], when the latitude is 42°, and the polar distance of the star 30°.

Solution. By means of proportional logarithms, and equations (392) and (393),

$\frac{1}{4}$. 1000ˢ = 4ᵐ 10ˢ	Prop. log.	1.6355		1.6355
L = 42°	cotan.	10.0456	cos.	9.8711
30°	cotan.	10.2386		10.2386
corr. A = 130ˢ = 2ᵐ 10ˢ	Prop. log.	1.9197		
0.0664846				8.8227
corr. B = 48′ 41″			Prop. log.	0.5679

3. Calculate the corrections of Table C [B. p. 152] for the pole-

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star and the latitude of 30°, when the polar distance of this star is 1° 32′ 37″.

Solution. By (396) and (397),

	0.0664846	8.82273	8.82273
$a = 1000''$		3.00000	3.00000
$p = 1^\circ\ 32'\ 37''$	cosec.	11.56964	11.56964
$p + L = 31^\circ\ 32'\ 37''$	cos.	9.93056	
$p - L = -28^\circ\ 27'\ 23''$			9.94407
corr. C upper trans. $= 2103^s$		3.32293	
corr. C lower trans. $= 2170^s$			3.33644

4. An observer in Boston in the year 1840, wishing to determine his meridian line, observed three successive transits of β Cephei over the central vertical wire of his transit instrument, by means of a clock regulated to solar time, and found them to occur as follows; the first upper transit at $7^h\ 45^m\ 28^s$ P. M., the next inferior transit the next day at $7^h\ 41^m$ A. M., the third transit at $7^h\ 41^m\ 32^s$ P. M. What were the times of the star's passing the meridian the second day? and what was the azimuth error in the position of the instrument?

Solution.

The first interval $= 19^h\ 41^m - 7^h\ 45^m\ 28^s = 11^h\ 55^m\ 32^s$

The second interval $= 19^h\ 41^m\ 32^s - 7^h\ 41^m = 12^h\ \ 0^m\ 32^s$.

Hence $\delta\, i = 5^m = 300^s$.

Now $L = 42^\circ\ 21'$, $D = 69^\circ\ 52'$, $p = 20^\circ\ 8'$.

Hence, by Tables A and B

$$\text{corr. } A = 83^s \times 0.3 = 25^s,$$

$$\text{corr. } B = 31'\ 6'' \times 0.3 = 9'\ 19'';$$

so that the error in the time of the upper transit is

$$\tfrac{1}{4} \,.\, 300^s - 25^s = 75^s - 25^s = 50^s,$$

and the error in the time of the lower transit is

$$\tfrac{1}{4} \,.\, 300^s + 25^s = 75^s + 25^s = 100^s = 1^m\ 40^s.$$

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The times of the star's passing the meridian the second day were, then,

$$7^h\,41^m + 1^m\,40^s = 7^h\,42^m\,40^s \text{ A. M.}$$

and
$$7^h\,41^m\,32^s - 50^s = 7^h\,40^m\,42^s \text{ P. M.}$$

The error in the azimuth of the instrument was $9'\,19''$ to the west of north.

5. An observer at Boston, wishing to determine his meridian line, on the morning of January 1, 1840, observed, by means of a clock regulated to solar time, the superior transit of γ Ursæ Majoris at $5^h\,6^m\,54^s$ A. M., and the inferior transit of Polaris at $6^h\,12^m\,23^s$ A. M. What was the azimuth error in the position of the transit instrument?

Solution. The interval between these two transits is

$$6^h\,12^m\,23^s - 5^h\,6^m\,54^s = 1^h\,5^m\,29^s.$$

But, by the Nautical Almanac,

$12^h +$ R. A. of Polaris	$= 13^h\ \ 1^m\,59^s$
R. A. of γ Ursæ Majoris	$= 11^h\,45^m\,25^s$
Sideral Interval	$= \ 1^h\,16^m\,34^s$
Solar Interval	$= \ 1^h\,16^m\,22^s$
Observed Interval	$= \ 1^h\ \ 5^m\,29^s$
Error of Interval	$= \quad 10^m\,53^s = 653$.

Now for $1000''$ of azimuth error, and the latitude of Boston, Table C gives, since

Dec. of γ Ursæ Majoris . . .	$= 54°\,35'$
Error of lower trans. of Polaris . .	$= 1866^s$
Error of upper trans. of γ Ursæ Majoris .	$= \ \ 25^s$
Sum of errors	$= 1891^s$

Then the proportion

$$1891^s : 653^s = 1000'' : \text{azimuth error},$$

gives

$$\text{azimuth error} = 345'' = 5'\,45''\ \text{W.}$$

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6. An observer at Boston wishing to determine his meridian line, in the evening of December 17, 1839, observed by means of a clock regulated to solar time, the superior transit of α Cassiopeæ at $6^h\,48^m\,35^s$ P.M., and that of Polaris at $6^h\,53^m\,15^s$ P.M. What was the azimuth error in the position of the transit instrument?

Solution. By the Nautical Almanac,

R. A. of Polaris	$= 1^h\ \ 2^m\,26^s$
R. A. of α Cassiopeæ	$= 0^h\,31^m\,28^s$
Sideral Interval	$= 0^h\,30^m\,58^s$
Solar Interval	$= 0^h\,30^m\,53^s$
Observed Interval	$= 0^h\ \ 4^m\,40^s$
Error of Interval	$= 0^h\,26^m\,13^s = 1573^s$.

Now Table C gives, for $1000''$ of azimuth error and the latitude of Boston, since

Dec. of α Cassiopeæ	$= 55°\,40'$
Error of trans. of Polaris	$= 1777^s$
Error of trans. of α Cassiopeæ	$= 26^s$
Diff. of errors	$= 1751^s$

Then, the proportion

$$1751^s : 1573^s = 1000'' : \text{azimuth error}$$

gives

$$\text{azimuth error} = 900'' := 15'\,0''\text{ E.}$$

7. Calculate the proportional logarithm of $0°\,2'\,33''$.

Ans. 1.8487.

8. Calculate the proportional logarithm of $2°\,59'\,12''$.

Ans. 0.0019.

9. Calculate the corrections of Tables A and B, when the latitude is $54°$, and the star's polar distance $20°$.

Ans. Corr. A $= 125^s$.
Corr. B $= 38'\,48''$.

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10. Calculate the corrections of Table C, when the latitude is $20°$, and the polar distance $5°$.

Ans. For the upper transit, corr. C $= 691^s$.
For the lower transit, corr. C $= 737^s$.

11. An observer at Boston, in the year 1840, wishing to determine his meridian line, observed three successive transits of Polaris, by means of a clock regulated to solar time. The first lower transit was observed at 6^h A. M., the next transit at $6^h\ 2^m\ 11^s$ P. M., and the second lower transit at $5^h\ 56^m\ 4^s$ A. M. What was the time of the star's passing the meridian the second morning? and what was the azimuth error in the position of the instrument?

Ans. The time of the third merid. trans. was $5^h\ 58^m\ 49^s$ A. M.
The azimuth error $= 15'\ 27''$ W.

12. An observer at Boston, wishing to determine his meridian line by means of a clock regulated to solar time, observed the inferior transit of Polaris on April 4, 1839, at 0^h A. M., and the superior transit of η Ursæ Majoris at $0_h\ 53^m\ 59^s$ A. M. What was the azimuth error in the position of his transit instrument?

The R. A. of Polaris is $1^h\ 0^m\ 50^s$, that of η Ursæ Majoris is $13^h\ 41^m\ 14^s$, and the declination of η Ursæ Majoris is $50°\ 7'$ N.

Ans. The azimuth error $= 7'\ 18''$ W.

13. An observer at Boston, wishing to determine his meridian line, in the evening of May 1, 1839, observed, by means of a clock regulated to solar time, the lower transit of Polaris at $9^h\ 49^m\ 22^s$ P. M., and that of α Cassiopeæ at $9^h\ 52^m$ P. M. What was the azimuth error of the instrument?

The R. A. of Polaris	$= 1^h\ \ 0^m\ 56^s$.
The R. A. of α Cassiopeæ	$= 0^h\ 31^m\ 22^s$.
The Dec. of α Cassiopeæ	$= 55°\ 39'$ N.

Ans. The azimuth error $= 18'\ 34''$ W.

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CHAPTER IV.

LATITUDE.

46. *Problem. To find the latitude of a place.*

Solution. The latitude of the place is evidently, from (fig. 34), equal to the altitude of the pole; so that this problem is the same as to find the altitude of the pole, which would be done without difficulty if the pole were a visible point of the celestial sphere.

First Method. By Meridian Altitudes. [B. p. 166–175.]

Observe the altitude of a star at its transit over the meridian, and let

A = the altitude of the star,

A' = $*$'s dist. from point of horizon below the pole;

then, if the notation of § 28 is used, it is evident, from (fig. 34), that

$$L = A' \mp p; \tag{403}$$

the upper sign being used when the transit is a superior one, and the lower sign when it is an inferior one.

I. Suppose the observed transit to be a superior one; then, if it passes upon the side of the zenith opposite to the pole, we have

$$A' = 180° - A, \ p = 90° \mp D,$$

and (403) becomes

$$L = 90° - (A \mp D) = (90° - A) \pm D = z \pm D; \tag{404}$$

the upper sign being used when the declination and latitude are of the same name, and the lower sign when they are of different names.

But if the star passes upon the same side of the zenith with the pole, we have

$$A' = A, \ p = 90° - D,$$

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and (403) becomes

$$L = (A + D) - 90^\circ = D - (90^\circ - A) = D - z. \qquad (405)$$

II. If the transit is an inferior one, we have

$$A' = A, \;\; p = 90^\circ - D,$$

and (403) becomes

$$L = (A - D) + 90^\circ = A + (90^\circ - D). \qquad (406)$$

Equations (404) and (405) agree with the rule of Case I, [B. p. 166], and (406) with Case II, [B. p. 167.]

III. If both transits are observed, and if A' and A are referred to the upper transits, and

$$A_1 = \text{the altitude at the lower transit,}$$

we have, by (403),

$$L = A' - p$$

$$L = A_1 + p,$$

the sum of which is

$$L = \tfrac{1}{2}(A' + A_1); \qquad (407)$$

so that the latitude is determined in this case without knowing the star's declination.

Second Method. By a Single Altitude.

Observe the altitude and the time of the observation.

I. If the star is considerably distant from the meridian, we have given in the triangle PBZ (fig. 35), PB, BZ, and BPZ to find PZ, which may be solved by Sph. Trig. § 60, and gives by the notation of § 28,

$$\text{tang. } PC = \cos. h \text{ tang. } p = \pm \cos. h \text{ cotan. } D \qquad (408)$$

$$\cos. ZC = \cos. PC \,.\, \cos. z \sec. p$$

$$= \pm \cos. PC \,.\, \cos. z \text{ cosec. } D, \qquad (409)$$

in which the upper sign is used if the declination and latitude are of the same name, otherwise the lower sign.

$$90^\circ - L = PZ = PC \pm ZC$$

$$L = 90^\circ - (PC \pm ZC); \qquad (410)$$

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in which both signs may be used if they give values of L contained between $0°$ and $90°$, and in this case other data must be resorted to, in order to determine which is the true value of L.

Scholium. The problem is, by Sph. Trig. § 61, impossible, if the altitude is greater than the declination, when the hour angle is more than six hours.

II. If the latitude is known within a few miles, it may be exactly calculated by means of (376), or

$$\cos. z = \cos. [90° - (L + p)] - 2 \cos. L \cos. D (\sin. \tfrac{1}{2} h)^2. \quad (411)$$

But if A is the star's observed altitude, and A_1 its meridian altitude at its upper transit, (403) gives

$$A_1 = L + p, \text{ or } = 180° - (L + p),$$

and (411) becomes, by transposition,

$$\sin. A_1 = \sin. A + 2 \cos. L \cos. D (\sin. \tfrac{1}{2} h)^2; \quad (412)$$

from which the meridian altitude may be calculated by means of Table XXIII, as in the Rule. [B. p. 200.]

III. A formula can also be obtained from (340), which is particularly valuable when the star is, as it always should be in these observations, near the meridian.

In this case we have in (340) applied to PBZ

$$2 s = 90° - L + p + x = 180° - L + p - A \quad (413)$$

$$2 s - 2 PZ = L + p - A$$

$$= A_1 - A \text{ or } = 180° - (A_1 + A) \quad (414)$$

$$2 s - 2 PB = 180° - L - p - A$$

$$= 180° - (A_1 + A) \text{ or } = A_1 - A; \quad (415)$$

and if these values are substituted in (340), after it is squared and freed from fractions, they give

$$(\sin. \tfrac{1}{2} h)^2 \cos. L \cos. D = \sin. \tfrac{1}{2} (A_1 - A) \cos. \tfrac{1}{2} (A_1 + A), \quad (416)$$

or

$$\sin. \tfrac{1}{2} (A_1 - A) = (\sin. \tfrac{1}{2} h)^2 \cos. L \cos. D \sec. \tfrac{1}{2} (A_1 + A); \quad (417)$$

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and if, in the second member of this equation, the value of A_1 is used, which is obtained from the approximate value of the latitude, the difference between the observed and the meridian altitudes may be found at once; and this difference is to be added to the observed altitude to obtain the meridian altitude.

IV. If the star is very near the meridian, $\frac{1}{2}(A_1 - A)$ and $\frac{1}{2}h$ will be so small, that we may put

$$\sin. \tfrac{1}{2}(A_1 - A) = \tfrac{1}{2}(A' - A) \sin. 1''$$
$$\sin. \tfrac{1}{2}h = \tfrac{1}{2}h \sin. 1^s = \tfrac{15}{2}h \sin. 1'';$$

which, substituted in (417) give, by supposing A_1 equal to A in the second member, which is very nearly the case,

$$A_1 - A = \tfrac{15}{2}h^2 \sin. 1^s \cos. L \cos. D \sec. A_1. \qquad (418)$$

This value of $A_1 - A$ is proportional to h^2, so that if it were calculated for

$$h = 1^s,$$

any other value might be calculated by multiplying by h^2. Now Table XXXII, of the Navigator, contains the values of $A_1 - A$ for all latitudes and for all declinations less than 24°, excepting a few latitudes in which the meridian transit of the observed body is too near the zenith for this observation to be accurate; and Table XXXIII contains all the values of h^2, where h is less than 13^m.

V. If the observed star is very near the pole, we have in (408)

$$\text{tang. } PC = \cos. h \text{ tang. } p; \qquad (419)$$

so that as p is very small, PC must be likewise small, and we have

$$\cos. h = \frac{\text{tang. } PC}{\text{tang. } p} = \frac{PC}{p}$$
$$PC = p \cos. h; \qquad (420)$$

and, by Pl. Trig. § 22,

$$\cos. PC = 1, \ \sin. D = \cos. p = 1,$$

whence, by (409), and (410),

$$\cos. ZC = \cos. z, \ ZC = z,$$
$$L = 90° - PC - ZC = 90° - z - PC$$
$$= A - p \cos. h; \qquad (421)$$

so that p cos. h may be regarded as a correction to be subtracted from A when it is positive, that is, when the hour angle is less than 6 hours, or greater than 18 hours; and it is to be added when the hour angle is greater than 6 hours and less than 18 hours.

The table [B. p. 206] for the pole star was calculated for the year 1840, when

$$\text{its R. A.} = 1^h\ 2^m; \text{ its dec.} = 88^\circ\ 27' \text{ nearly.}$$

VI. The method of determining the latitude by means of the pole star is so accurate in practice, that tables are given in the Nautical Almanac for correcting the observed altitude for differences of latitude, and for changes in the right ascension and declination of the star. Of these corrections the *first* is the same as that of the Navigator, and is computed from (421) by using the pole star's mean right ascension and declination for the year; and the *third* is the correction for the change in the star's right ascension and declination. Both of these corrections may, however, be full as readily obtained by direct computation from (421), if the actual right ascension and declination of the star are at once substituted in the formula. The *second* correction of the Nautical Almanac arises from the error in supposing ZC to be equal to z, and is so small that the mean right ascension and declination of the pole star may be used in its computation.

We have then, in the right triangle BPC, since p and BC are small,

$$\frac{BC}{p} = \frac{\sin. BC}{\sin. p} = \sin. h,$$

or

$$BC = p \sin. h;$$

and the right triangle BCZ gives, since

$$BZ = 90^\circ - A$$

$$CZ = 90^\circ - PC - L = 90^\circ - p \cos. h - L$$

$$\cos. BZ = \cos. CZ \cos. BC$$

$$\sin. A = \sin. (p \cos. h + L) \cos. BC$$

$$\sin. (p \cos. h + L) - \sin. A = \sin. (p \cos. h + L)\,(1 - \cos. BC)$$

$$2 \cos. \tfrac{1}{2}(p \cos. h + L + A) \sin. \tfrac{1}{2}(p \cos. h + L - A)$$

$$= 2 \sin. (p \cos. h + L) \sin.^2 \tfrac{1}{2} BC;$$

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or, since A differs but little from L, and p and BC are small,

$$\cos. L . (p \cos. h + L - A) \sin. 1'' = \tfrac{1}{2} \sin. L . (BC)^2 \sin.^2 . 1''$$

$$p \cos. h + L - A = \tfrac{1}{2} p^2 \tan. L . \sin.^2 . h \sin. 1'', \qquad (422)$$

which gives the required second correction, and this method of computing the latitude is most accurate when h is nearly 6 or 18 hours.

VII. The formula (417) may, however, be used directly for observations of the pole star more readily than the tables of the Nautical Almanac, and gives at once

$$L = A p + p (\sin. \tfrac{1}{2} h)^2 \cos. L \sec. \tfrac{1}{2} (A + L + p), \qquad (423)$$

and is most accurate when h is small.

VIII. By applying (417) to the lower transit of the pole star, that is, substituting its supplement for h, and making

$$A_1 = L - p,$$

it becomes

$$L = A + p - p (\cos. \tfrac{1}{2} h)^2 \cos. L \sec. \tfrac{1}{2} (A + M - p), \qquad (424)$$

which is most accurate when h is nearly 12 hours.

Third Method. By Circummeridian Altitudes.

I. If several altitudes are observed near the meridian, each observation may be reduced separately by (417) and (418), and the mean of the resulting latitudes is the correct latitude.

II. But if (418) is used, the mean of the values of $A_1 - A$ is evidently obtained by multiplying the mean of the values of h^2 by the constant factor; and if to the mean of the values of $A_1 - A$, the mean of the values of A is added, the sum is the mean of the value of A_1, whence precisely the same mean of resulting latitude is obtained as by the former method, but with much less calculation.

III. If the star is changing its declination in the course of the observations, this change may, in all cases which can occur if the hour angle is small, be neglected in the value of cos. D. But the value of A_1 will not, in this case, be at each observation equal to the meridian altitude, but will differ from it by the difference of the star's declination. Let the change of the star's declination in one minute

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be denoted by δD, which is positive when the star is approaching the elevated pole; and if h is the star's hour angle at the time of observation, which is negative before the star arrives at the meridian and afterwards positive, the whole change of declination is $h \delta D$, so that the correct meridian altitude is

$$A_1 - h \delta D.$$

The mean of the values of the corrected meridian altitude is, therefore, equal to the mean of the values of A_1 diminished by the mean of the values of $h \delta D$; and, if H denotes the mean of the hour angles h (regard being had to their signs), the correct meridian altitude is the mean of the values of A_1 diminished by $H \delta D$.

Fourth Method. By Double Altitudes.

I. Let two altitudes of a star, which does not change its declination, be observed, and the intervening time. Then (fig. 39) let Z be the zenith, P the pole, S and S' the positions of the star; join ZS, ZS', PS, PS', and $SS'M$; draw PT to the middle T of SS', join ZT, and draw ZV perpendicular to PT. Let

$$p = PS = PS' = 90° - D, \; SPS' = \text{elapsed time} = h,$$
$$ST = A = S'T, \; PT = 90° - B$$
$$A_1 = 90° - ZS, \; A'_1 = 90° - ZS'$$
$$ZTP = T, \; ZT = F, \; ZV = C$$
$$TV = Z, \; PV = 90° - E;$$

in which D and B are positive, when the latitude and declination are of the same name, but negative, if they are of contrary names; Z is positive, if the zenith is nearer the elevated pole than the point M.

Now the triangle TPS gives

$$\sin. A = \sin. PS \sin. SPT = \cos. D \sin. \tfrac{1}{2} h$$

$$\cos. PS = \cos. PT \cos. A, \text{ or } \sin. D = \sin. B \cos. A, \tag{425}$$

or

$$\text{cosec. } A = \sec. D \text{ cosec. } \tfrac{1}{2} h \tag{426}$$

$$\text{cosec. } B = \cos. A \text{ cosec. } D. \tag{427}$$

The triangles ZTS and ZTS' give

$$\sin. A_1 = \cos. F \cos. A - \sin. F . \sin. A \sin. T, \tag{428}$$

$$\sin. A'_1 = \cos. F \cos. A + \sin. F . \sin. A \sin. T. \tag{429}$$

The sum and difference of which is, by (43) and (44),

$$\sin.\tfrac{1}{2}(A_1+A'_1)\cos.\tfrac{1}{2}(A'_1-A_1)=\cos.F\cos.A, \qquad (430)$$

$$\sin.\tfrac{1}{2}(A'_1-A_1)\cos.\tfrac{1}{2}(A_1+A'_1)=\sin.F\sin.A\sin.T. \qquad (431)$$

But triangle ZTV gives

$$\sin.C=\sin.F\sin.T, \qquad (432)$$

$$\cos.F=\cos.C\cos.Z; \qquad (433)$$

which, substituted in (430) and (431), give

$$\sin.C=\sin.\tfrac{1}{2}(A'_1-A_1)\cos.\tfrac{1}{2}(A_1+A'_1)\,\text{cosec.}A, \qquad (434)$$

$$\sec.Z=\cos.A\cos.C\sec.\tfrac{1}{2}(A_1-A'_1)\,\text{cosec.}\tfrac{1}{2}(A'_1+A_1). \qquad (435)$$

But $$PV=PT-TV,$$

or $$90^\circ-E=90^\circ-B-Z$$

$$E=B+Z. \qquad (436)$$

Lastly, triangle ZPV gives

$$\cos.PZ=\cos.ZV\cos.PV$$

$$\sin.L=\cos.C\sin.E. \qquad (437)$$

Equations (426, 427, 434–437) correspond to the rule and formula given in the Navigator. [B. p. 180.]

II. Another method of calculating the values of B, C, and Z, has been given, which dispenses with A and one opening of the tables, and may therefore be preferred by some computers, although it requires one more logarithm. Triangle TPS gives

$$\text{tang.}PT=\cos.\tfrac{1}{2}h\,\text{tang.}PS,$$

or $$\text{cotan.}B=\cos.\tfrac{1}{2}h\,\text{cotan.}D. \qquad (438)$$

The substitution of (426) in (434) gives

$$\sin.C=\cos.\tfrac{1}{2}(A_1+A'_1)\sin.\tfrac{1}{2}(A'_1-A_1)\sec.D\,\text{cosec.}\tfrac{1}{2}h. \qquad (439)$$

Triangle PTS gives

$$\cos.A=\sin.D\,\text{cosec.}B; \qquad (440)$$

which, substituted in (435), gives (441)

$$\sec.Z=\cos.C.\sin.D\,\text{cosec.}B\,\text{cosec.}\tfrac{1}{2}(A_1+A'_1)\sec\tfrac{1}{2}(A'_1-A_1).$$

Corollary. The hour angle ZPT is the mean between the hour angles ZPS and ZPS, and if we put

$$ZPT = H,$$

the triangle ZPV gives

$$\text{tang. } H = \text{tang. } C \text{ sec. } E, \qquad (442)$$

as in [B. p. 181.]

III. When the latitude is known within a few miles. In this case let

$$L' = \text{the assumed latitude},$$

and the triangle ZPV gives

$$\text{sin. } C = \text{cos. } L' \text{ sin. } H; \qquad (443)$$

whence, by (439), (444)

$$\text{sin. } H = \text{cos. } \tfrac{1}{2}\,(A_1 + A_1') \text{ sin. } \tfrac{1}{2}\,(A_1' - A_1) \text{ sec. } L' \text{ sec. } D \text{ cosec. } \tfrac{1}{2}\,h.$$

$$ZPS' = H - \tfrac{1}{2}\,h, \qquad (445)$$

whence the hour angle ZPS', corresponding to the observation at S', is known, and the latitude may be found by the method of a single altitude.

IV. *Douwes's Method.* Formula (444) is, by (44),

$$2 \text{ sin. } H = (\text{sin. } A_1' - \text{sin. } A_1) \text{ sec. } L' \text{ sec. } D \text{ cosec. } \tfrac{1}{2}\,h. \qquad (446)$$

The combination of the formulas (446, 445), and the method of computing the latitude by a single altitude, corresponds exactly to the rule given in the Navigator. [B. p. 185.]

The log. cosec. $\frac{1}{2}\,h$ is not only given in Table XXVII, but also in Table XXIII, where it is called the log. $\frac{1}{2}$ elapsed time of $\frac{1}{2}\,h$.

The value of

$$\begin{aligned} \text{log. } 2 \text{ sin. } H - 5 &= \text{log. sin. } H + \text{log. } 2 - 5 \\ &= \text{log. sin. } H - \text{ar. co. log. } 2 + 5 \\ &= \text{log. sin. } H - 4.69897 \\ &= 5.30103 - \text{log. elapsed time of } H \qquad (447) \end{aligned}$$

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is inserted in Table XXIII, and is called the log. middle time of H. The 5 is subtracted from log. 2 sin. H, on account of the different values of the radius in Tables XXIV and XXVII.

Scholium. When the calculated latitudes differ much from the assumed latitude, the calculation must be gone over again, with the calculated latitude instead of the assumed latitude. This labor may be avoided by noticing, in the course of the original calculation, the difference which would arise from a change of 10′ in the value of the assumed latitude, and calculating the correction of the latitude by the rule of double position. The error of the hypothesis is, in each case, the excess of the calculated above the assumed latitude, and the proportion is

$$\text{diff. of errors : diff. of hyp.} = \text{least error : corr. of hyp.} \tag{448}$$

V. If the star has increased its declination a little during the interval between the observations, the second altitude will also be increased, and will require a reduction, before applying either of these methods, in which the declination is supposed to be unchanged; or else the first declination and the first altitude must be increased.

Thus if $S\,a$ is the increase of declination, and if $a\,b$ is drawn perpendicular to ZS, $S\,b$ will be the increase of altitude. By putting

$$S\,a = \delta\,D,\ S\,b = \delta\,A,$$

we have

$$\delta\,A = \cos.\,S\,.\,\delta\,D, \tag{449}$$

or, from the triangle ZSP,

$$\delta\,A = \frac{\sin.\,L - \sin.\,A_1 \sin.\,D}{\cos.\,A_1 \cos.\,D}\,.\,\delta\,D, \tag{450}$$

and, by (41) and (42),

$$\delta\,A_1 = \frac{2 \sin.\,L - \cos.\,(A_1 - D) + \cos.\,(A_1 + D)}{\cos.\,(A_1 - D) + \cos.\,(A_1 + D)}\,\delta\,D, \tag{451}$$

in which D is to be negative, when the latitude and declination are of contrary names. Hence the value of $\delta\,A_1$ can be computed by this formula, and it is to be added to the first altitude when the declination is increasing, and subtracted when the declination is decreasing. Since the value of $\delta\,A_1$ is proportional to $\delta\,D$, it may be computed for some assumed value of $\delta\,D$, and arranged in a table like Table

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XLVI of the Navigator, and the value of δA_1 can be computed from this table by a simple proportion. The rest of the calculation can be conducted according to the preceding methods, as in [B. p. 189.]

VI. If two stars are observed, whose declinations are quite different. Then, if P (fig. 40) is the pole, Z the zenith, S and S' the places of the stars.

$$A_1 = 90^\circ - ZS = \text{the less altitude,}$$

$$A_1' = 90^\circ - ZS' = \text{the greater altitude,}$$

$$D = 90^\circ - PS = \text{the declination of star at } S,$$

$$D' = 90^\circ - PS' = \text{the declination of star at } S',$$

$$H = SPS' = \text{hour angle} = \text{interv. of sideral time.}$$

Then, in the triangle PSS', PS, PS', and H, are given to find

$$SS' = C, \text{ and } S'SP = 90^\circ - F.$$

Next, in the triangle ZSS', the three sides are known, to find the angle

$$ZSS' = Z.$$

Hence $$ZSP = 90^\circ - G = 90^\circ - F - Z$$

$$G = F + Z.$$

Lastly, in the triangles ZSP, ZS, SP, and the included angle ZSP are given to find

$$ZP = 90^\circ - L.$$

This solution is precisely similar to the Rule in [B. p. 193]; and it is easy to prove the rules for the signs which are there given.

VII. If the distance SS' were observed, the angles ZSS' and $S'SP$ might be found from the triangles ZSS' and $S'SP$, in which the sides are all known, and the rest of the calculation would be as in the last method, and this method corresponds exactly to the Rule in [B. p. 197.]

47. Examples.

1. The correct meridian altitude of Aldebaran was found by

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observation, in the year 1838, to be 55° 45′, when its bearing was south; what was the latitude?

Solution.	The zenith distance	= 34° 15′ N.
	The declination	= 16° 11′ N.
	The latitude	= 50° 26′ N.

2. The correct meridian altitude of Canopus was found by observation, in the year 1839, to be 16° 25′, when its bearing was south; what was the latitude?

Solution.	The zenith distance	= 73° 35′ N.
	The declination	= 52° 36′ S.
	The latitude	= 20° 59′ N.

3. The correct meridian altitude of Dubhe was found by observation, in the year 1830, to be 50° 45′, when its bearing was north; what was the latitude?

Solution.	The zenith distance	= 39° 15′ S.
	The declination	= 62° 40′ N.
	The latitude	= 23° 25′ N.

4. If the correct meridian altitude of Dubhe, at its greatest elevation, were found by observation, in the year 1830, to be 50° 45′, when its bearing was south; what would be the latitude?

Solution.	The zenith distance	= 39° 15′ N.
	The declination	= 62° 40′ N.
	The latitude	= 107° 55′ N.

The problem is impossible.

5. The correct meridian altitude of Dubhe, at its least elevation, was found by observation, in the year 1830, to be 50° 45′; what was the latitude?

Solution.	The polar distance	= 27° 20′
	The altitude	= 50° 45′
	The latitude	= 78° 5′ N.

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6. The correct meridian altitudes of Dubhe, at its greatest and least elevations, which were on opposite sides of the zenith, were found by observation to be $72° 4'$ and $53° 16'$; what was the latitude?

Solution.

	The greatest altitude	=	$72° \ 4'$
	The least altitude	=	$53° \ 16'$
	Diff. of altitudes	=	$18° \ 48'$
$180° -$	Diff. of altitudes	=	$161° \ 12'$
	Latitude	=	$80° \ 36'$

7. The correct meridian altitudes of Dubhe, at its greatest and least altitudes, which were on the same side of the zenith, were found by observation to be $15° 1'$ and $69° 41'$; what was the latitude?

Solution.

Greatest alt.	=	$69° \ 41'$
Least alt.	=	$15° \ \ 1'$
Sum of alts.	=	$84 \ \ 42'$
Latitude	=	$42° \ 21'$ N.

8. In a northern latitude, the altitude of Aldebaran was found by observation, in the year 1839, to be $25° 38'$, when its hour angle was $4^h \ 12^m \ 20^s$; what was the latitude?

Solution. By (408, 409, 410),

$h = 4^h \ 12^m \ 20^s$	cos. 9.65580	
$D = 16° \ 11'$	cotan. 10.53729	cosec. 10.55484
$90° - PC = 32° \ 40'$	cotan. 10.19309	sin. 9.73215
	$A = 25° \ 38'$	sin. 9.63610
$ZC = 33° \ \ 6'$		cos. 9.92309
$L = 65° \ 46'$ N.		

9. In lat. $65° 40'$ N. nearly, the latitude of Aldebaran was found by observation, in the year 1839, to be $25° 38'$, when its hour angle was $4^h \ 12^m \ 20^s$; what was the true latitude?

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Solution. I. By (412),

	$65^\circ\ 40'$	cos.	9.61494
	$16^\circ\ 11'$	cos.	9.98244
	$4^h\ 12^m\ 20^s$	log. Ris.	4.73823
	Nat. num. 21657		4.33561
$25^\circ\ 38'$	Nat. sin. 43261		
$49^\circ\ 31'$ N.	Nat. cos. 64918		
$16^\circ\ 11'$ N.			
$65^\circ\ 42'$ N. = the latitude.			

Had the assumed latitude been taken $10'$ more, the calculated latitude would have been $65^\circ\ 48\frac{1}{2}'$ N.; hence, by (448),

$$3\tfrac{1}{2} : 1\tfrac{1}{2} = 10' : 4' = \text{corr. of second hypothesis,}$$

or the latitude $= 65^\circ\ 46'$ N., as in the preceding example.

II. By (417),

$$\tfrac{1}{2}h = 2^h\ 6^m\ 10^s\ 2 \quad \text{log. sin. } 9.43720$$
$$L = 65^\circ\ 40' \quad \text{cos. } 9.61494$$
$$D = 16^\circ\ 11' \quad \text{cos. } 9.98244$$
$$A_1 = 40^\circ\ 31'$$
$$A = 25^\circ\ 38' \qquad A = 25^\circ\ 38'$$
$$A_1 A - = 14^\circ\ 51' \quad \tfrac{1}{2}(A_1 + A) = 33^\circ\ 4\tfrac{1}{2}' \quad \text{sec. } 10.07678$$
$$A_1 = 40^\circ\ 29' \quad \tfrac{1}{2}(A_1 - A) = 7^\circ\ 25\tfrac{1}{2}' \quad \text{sin. } 9.11136$$
$$\text{corr. } A_1 = \quad 2' = \text{corr. lat.}$$
$$\text{Lat.} = 65^\circ\ 40' + 2' = 65^\circ\ 42' \text{ as before.}$$

10. Calculate the variation of a star's altitude in one minute from the meridian, when the declination is 12° N., and the latitude 5° N.

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Solution. If $A_1 - A$ is required in seconds, (418) gives

$$A_1 - A = 450 \sin. 1^m \cos. L \cos. D \sec. A_1$$

$$\begin{aligned}
\log. 450 \sin. 1^m &= \log. 450 + \log. \sin. 1^m \\
&= 2.65321 + 7.63982 \quad = \quad 0.29303 \\
D &= 12^\circ \qquad\qquad \cos. \ 9.99040 \\
L &= \ \ 5^\circ \qquad\qquad \cos. \ 9.99834 \\
A_1 &= 83^\circ \qquad\qquad \sec. \ 0.91411 \\
A_1 - A &= 15''.7, \text{ as in Table XXXII.} \ 1.19588
\end{aligned}$$

11. Calculate the tabular number for $11^m\ 48^s$ in Table XXXIII.

Solution.

$11^m\ 48^s = 708^s$	log.	2.85003
60^s	log.	1.77815
		1.07188
		2
139.2, as in Table XXXIII.		2.14376

12. In lat. 45° 28′ N. nearly, the correct altitude of Aldebaran was found by observation, in the year 1839, to be 60° 40′ 20″, when its hour angle was $7^m\ 17^s$. What was the true latitude, if the declination of Aldebaran was 16° 11′ 9″.2 N.?

Solution.

From Table XXXII	2″.7
From Table XXXIII	53
2′ 23″.1 =	143″.1
	60° 40′ 20″
Third alt. =	60° 42′ 43″.1
Dec. =	16° 11′ 9″.2
Lat. =	45° 28′ 26″.1 N.

13. In lat. 40° N. nearly, the sum of ten correct central altitudes of the sun, when its declination was 20° S. were 300° 6′ 40″. The hour angles of these observations were $4^m\ 15^s$, 3^m, $2^m\ 6^s$, $1^m\ 8^s$, 30^s, 50^s, $1^m\ 12^s$, $2^m\ 15^s$, $3^m\ 10^s$, $4^m\ 25^s$. What is the true latitude, if the change of declination is neglected?

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Solution. The numbers of Table XXXIII are

4^m	15^s	gives	18.1
3	0		9.0
2	6		4.4
1	8		1.3
0	30		0.2
0	50		0.7
1	12		1.4
2	15		5.1
3	10		10.0
4	25		19.5

Sum $= 69.7$

Mean $= 6.97$

Table XXXII gives $1''.6$

$11''$

Mean of observations $= 30° \; 0' \; 40''$

Merid. alt. $= 30° \; 0' \; 51''$

Dec. $= 20°$ S.

Lat. $= 39° \; 59' \; 9''$ N.

14. At Göttingen, in lat. 51° 32′ N. nearly, the correct central altitudes of the sun on the 11th of March, 1794, were by observation

34° 54′ 46″ when the hour angle was	$-\,9^m \; 41^s$
34 55 26	$-\,8 \;\; 19$
34 56 8	$-\,6 \;\; 39$
34 56 31	$-\,5 \;\; 16$
34 56 53	$-\,3 \;\; 49$
34 57 6	$-\,2 \;\; 47$
34 57 18	0 19
34 57 11	2 5
34 57 3	3 9
34 56 48	4 36
34 56 26	6 8

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The sun's meridian declination was $3° 30' 38''$ S., and it was decreasing at the rate of $0''.98$ in a minute. What is the true latitude?

Solution. The mean of the altitude is that of the numbers of Table XXXIII is $30''.0$; which, multiplied by $1''.5$ from Table XXXII, gives . . . . — $34° 56' 30''.5$; $45''.0$

The mean of the hour angles is, regarding their signs, $-1^m 50^s$, which, multiplied by $0''.98$, gives by (418), for the correction of the meridian altitude . . . — $1''.8$

The meridian altitude	=	$34° 57' 17''.3$
The declination	=	$3° 30' 38''$ S.
The latitude	=	$51° 32' 4''.7$ N.

which agrees exactly with the calculations of Littrow in his Astronomy.

15. Calculate the correction for the altitude of the pole star [B. p. 206], when the right ascension of the zenith is $2^h 7^m$.

Solution. By (421),

$h = 2^h 7^m - 1^h 2^m = 1^h 5^m$	sec.	0.0177
$p = 1° 33'$	Prop. log.	0.2868
Corr. alt. $= 1° 29'$, as in the table,	Prop. log.	0.3045

16. When the right ascension of the zenith was $7^h 9\frac{1}{2}^m$, the altitude of the pole star was observed at Newburyport to be $42° 44'$. What is the latitude of Newburyport?

Solution.

The correction of table	=	$0° 3'$
Altitude . . .	=	$42° 44'$
Latitude . . .	=	$42° 47'$

17. Calculate the log. elapsed time and log. middle time of Table XXIII for $3^h 7^m 10^s$.

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Solution. By Table XXVII and (447),

$3^h\ 7^m\ 10^s$	cosec.	$0.13635 =$ log. elapsed time	
		5.30103	
		$5.16368 =$ log. mid. time.	

18. Calculate the variation of the altitude of a star arising from the change of 100 seconds in the declination, when the latitude is $40°$, the declination $10°$, and the altitude $30°$.

Solution. By (451),

$L = 40°$, $2 \times$ Nat. sin.	1.2856	1.2856	
$A_1 - D = 20°$ Nat. cos. 0.9397	$-$ 0.9397	0.9397	
$A_1 + D = 40°$ Nat. cos. 0.7660	0.7660	$-$ 0.7660	
1.7057	1.1119	1.4593	

1.7057 (ar. co.)	9.7681	9.7681
$100'' \times 1.1119$	2.0461	
$100'' \times 1.4593$		2.1641
$65'' =$ var. when D is $+$,	1.8142	
$86'' =$ var. when D is $-$,		1.9322

19. The moon's correct central altitude was found, by observation, to be $53° 43'$, when her declination was $14° 16'$ N. After an interval, in which the hour angle was $1^h\ 44^m\ 15^s$, her correct central altitude was $42° 29'$, and her declination $13° 52'$ N. The latitude was $48° 50'$ nearly; what was it exactly?

Solution. Table XLVI gives, for the second alt. . $83''$

Whole change of declination . . . . $24'$

Correction of second altitude . . . . $20'$

Corrected second alt. $= 42° 49'$, dec. $= 14° 16'$ N.

I. By Bowditch's first method.

$1^h\ 44^m\ 15^s$	cosec. 0.64689		
14° 16′	sec. 0.01360	cosec. 0.60830	
A	cosec. 0.66049	cos. 9.98937	cos. 9.98937
B = 14° 38′ N.		cosec. 0.59767	
	cos. 9.82326	½ sum alts. = 48° 16′	cosec. 0.12712
	sin. 8.97762	½ diff. alts. = 5° 27′	sec. 0.00197
C	sin. 9.46137	cos. 9.98102	cos. 9.98102
Z = 37° 19′ N.			sec. 0.09948
E = 51° 57′ N.		sin. 9.89624	
Latitude = 48° 55′ N.		sin. 9.87726	

II. By the method (438–441).

$1^h\ 44^m\ 15^s$	cos. 9.98867	cosec. 0.64689	
14° 16′	cotan. 0.59469	sec. 0.01360	sin. 9.39170
B=14° 38′ N.	cotan. 0.58336		cosec. 0.59767
½ sum alts. = 48° 16′		cos. 9.82326	cosec. 0.12712
½ diff. alts. = 5° 27′		sin. 8.97762	sec. 0.00197
C	cos. 9.98102	sin. 9.46137	cos. 9.98102
Z=37° 17′ N.			sec. 0.09948
E=51° 57′ N.	sin. 9.89624		
Lat.=48° 55′ N.	sin. 9.87726		

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III. By Douwes's method.

			48° 50′ sec. 0.18161
53° 43′ N. sin.	80610		14° 16′ sec. 0.01360
42° 49′ N. sin.	67965		log. ratio 0.19521
	12645		log. 4.10192
$\frac{1}{2}$ ($1^h\ 44^m\ 15^s$) =	$52^m\ 7\frac{1}{2}^s$		log. el. time 0.64674
	$1^h\ 44^m\ 15\frac{1}{2}^s$		log. mid. time 4.94387
	$52^m\ 8^s$		log. ris. 3.41097
			log. ratio 0.19521
		1643	log. 3.21576
		80610	
34° 39½′ N. N. cos.		82253	
14° 16′ N.			
Lat.=48° 55½′ N.			

IV. By Bowditch's fourth method.

$1^h\ 44^m\ 15^s$	sec. 0.04657		tan. 9.68938
14° 16′ N.	tan. 9.40531	sin. 9.39170	
A=15° 48½′ S.	tan. 9.45188	cosec. 0.56485	cos. 9.98326
13° 52′ N.			
B= 1° 56½′ S.		cos. 9.99975	cosec. 1.47003
C=25° 16½′	cosec. 0.36961	cos. 9.95630	
		F = 4° 6½′ N.	cotan. 1.14367
53° 43′		Z = 51° 38′ N.	
		G = 55° 44½′ N.	sin. 9.91724
42° 29′	sec. 0.13225	sin. 9.82955	cotan. 0.03820
½sum=60° 44′	cos. 9.68920	I sec. 0.12938	tan. 9.95544
Rem.= 7° 1′	sin. 9.08692	K sin. 9.91823	I = 42° 4′ N.
	2) 19.27798	lat. sin. 9.87716	13° 52′ N.
½ Z = 25° 49′ N.	sin. 9.63899	lat. = 48° 54½′ N.	K = 55° 56′ N.

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20. The correct meridian altitude of Aldebaran was, by observation, $56° 25' 40''$ bearing south, and its declination at the time of the observation was $16° 8' 44''$ N.; what was the latitude?

Ans. $49° 43' 4''$ N.

21. The correct meridian altitude of Sirius was $70° 59' 33''$ bearing north, and its declination $16° 28' 9''$ S.; what was the latitude?

Ans. $35° 28' 36''$ S.

22. The meridian altitude of the sun's centre was $25° 38' 30''$ bearing south, and its declination $22° 18' 14''$ S.; what was the latitude?

Ans. $42° 3' 16''$ N.

23. The meridian altitude of the planet Jupiter was $50° 20' 8''$ bearing south, and its declination $18° 47' 37''$ N.; what was the latitude?

Ans. $58° 27' 29''$ N.

24. The altitude of the pole star was $30° 1' 30''$ below the pole, and its polar distance $1° 38' 2''$; what was the latitude?

Ans. $31° 39' 32''$ N.

25. The altitude of Capella on the meridian below the pole was $9° 52' 42''$, and its polar distance $44° 11' 33''$; what was the latitude?

Ans. $54° 4' 15''$ N.

26. The meridian altitude of the sun's centre was $7° 9' 11''$ below the pole, and its declination $23° 8' 17''$ N.; what was the latitude?

Ans. $74° 0' 54''$ N.

27. The two meridian altitudes of a northern circumpolar star were $61° 49' 13''$ and $47° 24' 27''$; what was the latitude?

Ans. $54° 36' 50''$ N.

28. In a northern latitude, the altitude of the sun's centre was $54° 9'$, when its hour angle was $32^m 40^s$, and its declination $11° 17'$ N.; what was the latitude?

Ans. $46° 27'$ N.

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29. In latitude 49° 15′ N. nearly, the altitude of the sun's centre was 14° 15′, when its hour angle was $1^h\ 40^m$, and its declination 23° 28′ S.; what was the true latitude?

Ans. 48° 55′ N.

30. Calculate the variation of a star's altitude in one minute from the meridian, when the declination is 3° and the latitude 7°.

Ans. It is 27″.9 when the dec. and lat. are of the same name, and 11″.2 when they are of contrary names.

31. Calculate the tabular number for $13^m\ 59^s$ in Table XXXIII.

Ans. 168.6.

32. In lat. 50° 30′ N. nearly, the altitude of Sirius was 22° 59′ 36″, when its hour angle was $4^m\ 15^s$, and its declination 16° 29′ 11″ S.; what was the true latitude?

Ans. 50° 30′ 49″ N.

33. In lat. 20° 27′ N. nearly, the sum of seven altitudes of Sirius was 371° 21′; the hour angles of the observations were 7^m, $5^m\ 3^s$, $2^m\ 12^s$, 9^s, 3^m, $4^m\ 6^s$, $8^m\ 13^s$; what was the true latitude, if the declination of Sirius was 16° 29′ 30″ S.?

Ans. 20° 26′ 18″ N.

34. In lat. 60° N. nearly, the sum of twelve central altitudes of the moon was 590°; the hour angles of the observations were $-9^m\ 3^s$, $-7^m\ 40^s$, $-6^m\ 12^s$, $-5^m\ 30^s$, $-3^m\ 2^s$, -1^m, -12^s, -50^s, $1^m\ 59^s$, 4^m, $7^m\ 30^s$, 10^m; the moon's meridian declination was 19° 0′ 58″.4 N., and her change of declination for one minute 13″.875; what was the true latitude?

Ans. 59° 50′ 2″.6 N.

35. Calculate the correction for the altitude of the pole star [B. p. 206], when the right ascension of the zenith is $9^h\ 7^m$.

Ans. 48′.

36. The altitude of the pole star was 25° 9′, when the right ascension of the zenith was $21^h\ 47^m$; what was the latitude?

Ans. 24° 8′ N.

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37. Calculate the log. elapsed time and log. middle time of Table XXIII for $5^h\ 58^m\ 10^s$.

Ans. Log. elapsed time $= 0.00001$
Log. middle time $= 5.30102$.

38. Calculate the variation of the altitude of a star arising from the change of 100 seconds in declination, when the latitude is 60°, the declination 20°, the altitude 30°, and the declination and latitude of the same name.

Ans. 85.″

39. Calculate the variation of the altitude of a star arising from the change of 100 seconds in declination, when the latitude is 50°, the declination 24°, and the altitude 20°.

Ans. It is 73″ when the lat. and dec. are of the same name, and 105″ when they are of contrary names.

40. The sun's correct central altitudes were found by observation to be 30° 13′ and 50° 4′; his declination was 20° 7′ N., and the interval of solar time between the observations was $2^h\ 55^m\ 32^s$; the assumed latitude was 56° 29′ N.; what was the true latitude?

Ans. 56° 47′ N.

41. The sun's correct central altitude was 41° 33′ 12″, his declination 14° N.; after an interval of $1^h\ 30^m$, his correct central altitude was 50° 1′ 12″, and declination 13° 58′ 38″ N.; the assumed latitude was 52° 5′ N.; what was the true latitude?

Ans. 52° 5′ N.

42. The moon's correct central altitude was 55° 38′, her declination 0° 20′ S.; after an interval in which the hour angle was $5^h\ 30^m\ 49^s$, her correct central altitude was 29° 57′, and her declination 1° 10′ N.; the assumed latitude was 23° 25′ S.; what was the true latitude?

Ans. 23° 24′ S.

43. The sun's correct central altitude was 16° 6′, his declination 8° 18′ N.; after an interval in which the hour angle was 3^h, his correct central altitude was 42° 14′ 9″, and his declination 8° 15′ N.; the assumed latitude was 49° N.; what was the true latitude?

Ans. 48° 50′ N.

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44. The moon's correct central altitude was $35° 21'$, and her declination $5° 31' 6''$ S.; after an interval in which the hour angle was $2^h 20^m$, her correct central altitude was $70° 1'$, and her declination $5° 28' 54''$ S.; the assumed latitude was $1° 30'$ S.; what was the true latitude?

Ans. $1° 33'$ S.

45. The altitude of Capella was $60° 45' 36''$, and her declination $45° 48' 21''$ N.; at the same instant, the altitude of Sirius was $17° 54' 12''$, and his declination $16° 28' 40''$ S.; the hour angle between the stars was $1^h 33^m 37^s$, and the latitude was about $53° 15'$ N.; what was the true latitude?

Ans. $53° 19'$ N.

46. The altitude of α Bootis was $50° 3' 39''$, and its declination $20° 10' 56''$ N.; the altitude of α Aquilæ was $41° 27'$, and its declination $8° 22' 35''$ N.; the difference of the hour angles of the observations was $5^h 35^m 5\frac{1}{2}^s$, and the assumed latitude $38° 27'$ N.; what was the true latitude?

Ans. $38° 28'$ N.

47. The distance of the centres of the sun and moon was found, by observation, to be $75°$; the sun's central altitude was $37° 40'$; the moon's central altitude was $55° 20'$; the sun's declination was $0° 17'$ S.; the moon's declination was $0° 36'$ N.; what was the latitude, supposing it to be north?

Ans. $23° 24'$ N.

48. The observer has been supposed stationary, in the preceding observations; but if he is in motion, his second altitude will differ from the altitude for this time at the first station, by the number of minutes by which the observer has approached the star or receded from it; so that the correction arising from this change of place is obviously computed by the method in [B. p. 183.]

49. In observing the meridian altitude of a star, the position of the meridian has been supposed to be known; but if it were not known, the meridian altitude can be distinguished from any other altitude from the fact that it is the greatest or the least altitude; so that it is only necessary to observe the greatest or the least altitude of the star.

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50. But if the star changes its declination, the greatest altitude ceases to be the meridian altitude. Let h denote the hour angle of the star at the time of observation. Then if the star did not change its declination, and if B were the number of seconds given by Table XXXII for the diminution of altitude in one minute from the meridian passage, $h^2 B$ would be the diminution of altitude in h minutes. But, since h is small, the altitude, at this time, is increased by the change of declination; so that if A is the number of minutes by which the star changes its declination in one hour, that is, the number of seconds by which it changes its declination in one minute, $h A$ will be the increase of altitude in the time of h, so that the altitude at the time h exceeds the meridian altitude by

$$h A - h^2 B. \tag{452}$$

If, then, h denotes the time of the greatest altitude, and $h + \delta h$ a time which differs very slightly from the greatest altitude; the greatest altitude exceeds the altitude at the time $h + \delta h$ by the quantity

$$\begin{aligned}(h A - h^2 B) &- [(h + \delta h) A - (h + \delta h)^2 B] \\ &= \delta h [(- A + 2 B h) + B \delta h], \end{aligned} \tag{453}$$

and δh can be supposed so small that $B \delta h$ may be insensible, and (453) becomes

$$\delta h (- A + 2 B h). \tag{454}$$

Now $- A + 2 B h$ cannot be negative, because h is supposed to correspond to the greatest altitude, and cannot be less than the altitude at the time $h + \delta h$. Neither can $- A + 2 B h$ be positive, for the altitude at the time h exceeds that at the time $h - \delta h$ by the quantity

$$- \delta h (- A + 2 B h),$$

which, in this case, would be negative, and the altitude at the time $h - \delta h$ would exceed the greatest altitude. Since, then, $- A + 2 B h$ can neither be greater nor less than zero, we must have

$$- A + 2 B h = 0$$

or

$$h = \frac{A}{2 B}, \tag{455}$$

and this value of h, substitued in (452), gives

$$\frac{A^2}{2\,B} - \frac{A^2}{4\,B} = \frac{A^2}{4\,B} \tag{456}$$

for the excess of the greatest altitude above the meridian altitude

51. If the observer were not at rest, his change of latitude will affect his observed greatest altitude in the same way in which it would be affected by an equal change in the declination of the star; so that the calculation of the correction on this account may be made by means of (455) and (456) precisely as in [B. p. 169.]

52. Examples.

1. An observer sailing N. N. W. 9 miles per hour, found, by observation, the greatest central altitude of the moon, bearing south, to be 54° 18′; what was the latitude, if the moon's declination was 6° 30′ S., and her increase of declination per hour 16′.52 ?

Solution. ☽'s zenith dist. = 35° 42′ N.
☽'s dec. = 6° 30′ S.

Approx. lat. = 29° 12′ N.
☽'s increase of dec. per hour = 16′.52 S.
Ship's change of lat. = 8′.3

$A = 24.82$, $A^2 = 616.0$
By Table XXXII $B = 2.9$, $4\,B = 11.6$
Corr. of gr. alt. = corr. of lat. = 52″ = 1′ nearly.
Lat. = 29° 12′ + 1′ = 29° 13′ N.

2. An observer sailing south $12\frac{1}{2}$ miles per hour, found, by observation, the greatest central altitude of the moon bearing south, to be 25° 15′; what was the latitude, if the moon's declination was 1° 12′ N., and her increase of declination per hour 18′.5 ?

Ans. 66° 1′ N.

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CHAPTER V.

THE ECLIPTIC.

53. The careful observation of the sun's motion shows this body to move nearly in the circumference of a great circle. This great circle is called *the ecliptic.* [B. p. 48.]

54. The angle which the ecliptic makes with the equator is called the *obliquity of the ecliptic.*

55. The points, where the ecliptic intersects the equator, are called the *equinoctial points;* or the *equinoxes.* The point through which the sun *ascends* from the southern to the northern side of the equator, is called the *vernal equinox;* and the other equinox is called the *autumnal equinox.*

The points 90° distant from the ecliptic are called the *solstitial points*, or the *solstices.* [B. p. 49.]

56. The circumference of the ecliptic is divided into twelve equal parts, called *signs*, beginning with the vernal equinox, and proceeding with the sun from west to east.

The names of these signs are *Aries* (♈), *Taurus* (♉), *Gemini* (♊), *Cancer* (♋), *Leo* (♌), *Virgo* (♍), *Libra* (♎), *Scorpio* (♏), *Sagittarius* (♐), *Capricornus* (♑), *Aquarius* (♒), *Pisces* (♓). The vernal equinox is therefore *the first point*, or beginning of Aries, and the autumnal equinox is the first point of Libra; the first six signs are north of the equator, and the last six south of the equator. The northern solstice is the first point of Cancer, and the southern solstice the first point of Capricorn. [B. p. 49.]

57. Secondary circles drawn perpendicular to the ecliptic are called *circles of latitude.*

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The circle of latitude drawn through the equinoxes is called *the equinoctial colure.*

The circle of latitude drawn through the solstices is called *the solstitial colure.* [B. p. 49.]

Corollary. The solstitial colure is also a secondary to the equator, so that it passes through the poles of both the equator and the ecliptic.

58. Small circles, drawn parallel to the equator through the solstitial points, are called *tropics.*

The northern tropic is called the *tropic of Cancer;* the southern tropic the *tropic of Capricorn.*

Small circles, drawn at the same distance from the poles which the tropics are from the equator, are called *polar circles.*

The northern polar circle is called the *arctic circle*, the southern the *antarctic.*

59. The *latitude of a star* is its distance from the ecliptic measured upon the circle of latitude, which passes through the star. If the observer is supposed to be at the earth, the latitude is called *geocentric latitude;* but if he is at the sun, it is *heliocentric latitude.* [B. p. 49.]

60. The *longitude of a star* is the arc of the ecliptic contained between the circle of latitude drawn through the star and the vernal equinox. [B. p. 50.]

Corollary. The longitude and right ascension of the first point of Cancer are each equal to 6^h, and those of the first point of Capricorn are each equal to 18^h.

61. The *nonagesimal point* of the ecliptic is the highest point at any time.

Corollary. The distance of the nonagesimal from the zenith is therefore equal to the distance of the zenith from the ecliptic, that is,

to the *celestial latitude of the zenith;* and the longitude of the nonagesimal is the *celestial longitude of the zenith.*

62. *Problem. To find the latitude and longitude of a star, when its right ascension and declination are known.*

Solution. Let P (fig. 35) be the north pole of the equator, Z the north pole of the ecliptic, and B the star. Then EQW will be the equator, $NESW$ the ecliptic, and $NPZS$ the solstitial colure, so that the point S is the southern solstice, and N the northern solstice. Now if the arc PB be produced to cut the equator at M, and ZB to cut the ecliptic at L; the angle ZPB is measured by the arc QM, that is, by the difference of the right ascensions of Q and M, or by the difference of the ✱'s right ascension and 18^h; that is,

$$ZPB = 18^h - \text{R. A.} = 24^h - (6^h + \text{R. A.})$$

or

$$= \text{R. A.} - 18^h = (\text{R. A.} + 6^h) - 24^h$$

or

$$= 24^h + \text{R. A.} - 18^h = \text{R. A.} + 6^h.$$

In the same way

$$PZB = NL = \text{Long.} - 90^\circ$$

or

$$= 360^\circ - (\text{Long.} - 90^\circ)$$

$$= -(\text{Long.} - 90^\circ),$$

in which the first values of ZPB and PZB correspond to the star's being east of the solstitial colure; the second and third values to the star's being west of the colure. We also have

$$PB = 90^\circ - \text{Dec.}$$

$$BZ = 90^\circ - \text{Lat.}$$

$$PZ = 90^\circ - ZQ = QS$$

$$= \text{obliquity of ecliptic} = \pm E, \quad (457)$$

in which the declination and latitude are positive when north, and negative when south, and E has the same sign with R. A. $- 12^h$.

The present problem does not, then, differ from that of § 28, and if we put

$$\pm A = PC - 90^\circ,$$

in which the upper sign is used, when R. A. — 12^h is positive, and otherwise the lower sign, we have by (298, 299, and 300),

$$\text{tang.}\ PC = \mp \text{cotan.}\ A = \text{cos.}\ (\text{R. A.} + 6^h)\ \text{cotan. Dec.}$$
$$= - \text{sin. R. A. cotan. Dec.} \qquad (458)$$

in which the signs are used as in the preceding equation; so that A and Dec. are always positive or negative at the same time. Instead of (458), its reciprocal may be used, which is

$$\mp \text{tang.}\ A = - \text{cosec. R. A. tang. Dec.} \qquad (459)$$

If, then,
$$B = E + A, \qquad (460)$$

we have

$$AP = \mp E - 90^\circ \mp A = \mp B - 90^\circ \qquad (461)$$

or
$$= 90^\circ \pm A \pm E = 90^\circ \pm B,$$

in which the upper or lower signs are used, as in (457). Hence

$$\text{cos.}\ PC : \text{cos.}\ AP = \mp \text{sin.}\ A : \mp \text{sin.}\ B = \text{sin.}\ A : \text{sin.}\ B$$
$$= \text{sin. Dec.} : \text{sin. Lat.} \qquad (462)$$

so that, since Dec. and A are both positive or both negative, B and Lat. must also be both positive or both negative. Again,

$$\text{sin.}\ PC : \text{sin.}\ PA = \text{cos.}\ A : \pm \text{cos.}\ B \qquad (463)$$
$$= \pm \text{cotan.}\ (\text{R. A} + 6^h) : \pm \text{cotan.}\ (\text{Long.} - 90^\circ)$$
$$= \pm \text{tang. R. A.} : \pm \text{tang. Long.}$$

in which the signs may be neglected, and Long. is to be found in the same quadrant with R. A., unless the foot P of the perpendicular falls within the triangle; in which case the first value of AP (461) is used, so that B is obtuse. In this case, the longitude is in the adjacent quadrant on the same side of the solstitial colure with the right ascension. These results agree with the Rule in [B. p. 145.]

63. *Corollary.* The latitude and longitude of the zenith, that is, the zenith distance and longitude of the nonagesimal, might be found by the same method. But another rule can be used, which is of peculiar advantage, where these quantities are often to be calculated

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for the same place. We have by (369) and (370), calling B the zenith, and putting

$$T = 24^h - ZPB \text{ or } = ZPB$$

$$F = \tfrac{1}{2}(PZB - ZBP) \text{ or } = 180^\circ - \tfrac{1}{2}(PZB - ZBP) \quad (464)$$

$$G = \tfrac{1}{2}(PZB + ZBP) \text{ or } = 180^\circ - \tfrac{1}{2}(PZB + ZPB) \quad (465)$$

$$\text{tang. } F = -\text{cosec. } \tfrac{1}{2}(PB + PZ) \text{ sin. } \tfrac{1}{2}(PB - PZ) \text{ cot. } \tfrac{1}{2} T$$
$$= \text{tang. } (24^h - F) \quad (466)$$

$$\text{tang. } G = -\text{sec. } \tfrac{1}{2}(PB + PZ) \cos. \tfrac{1}{2}(PB - PZ) \text{ cot. } \tfrac{1}{2} T \quad (467)$$

$$90^\circ + F + G = PZB + 90^\circ \text{ or } = 360^\circ - PZB + 90^\circ \quad (468)$$

$$= \text{Long. or } = 360^\circ + \text{Long.} \quad (469)$$

in which the first member of (466) is used when PB is greater than PZ, and the third when PB is less than PZ, that is, within the north polar circle; and the second members of (464, 465, 468) correspond to the position of the zenith at the east of the solstitial colure, but the third members to the west of the colure.

Again, by (354),

$$\text{tang. } \tfrac{1}{2}(90^\circ - \text{lat.}) = \text{tang. } \tfrac{1}{2} \text{ alt. nonagesimal}$$
$$= \cos. G \,.\, \text{sec. } F \text{ tang. } \tfrac{1}{2}(PB + PZ), \quad (470)$$

and the preceding formulas correspond to the rule in [B. p. 402.]

64. *Scholium.* The rule with regard to the values of G appears to be a little different, but the difference is only apparent; for it follows from (467), that G and $12^h - \tfrac{1}{2} T$ are, at the same time, both acute or both obtuse, unless

$$\tfrac{1}{2}(PB + PZ) > 90^\circ,$$

or

$$PB > 180^\circ - PZ, \quad (471)$$

which corresponds to the south polar circle.

65. The abridged method of calculating the altitude and longitude of the nonagesimal [B. p. 403], only consists in the previous computation of the values

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$$A = \text{log. } [\text{cos. } \tfrac{1}{2} (PB - PZ) \text{ sec. } \tfrac{1}{2} (PB + PZ)] \qquad (472)$$

$$C = \text{log. tang. } \tfrac{1}{2} (PB + PZ) \qquad (473)$$

$$B = \text{log. tang. } \tfrac{1}{2} (PB - PZ) - C \qquad (474)$$

$$= \text{log. } [\text{tang. } \tfrac{1}{2} (PB - PZ) \text{ cotan. } \tfrac{1}{2} (PB + PZ)]$$

$$= \text{log. } [\text{cosec. } \tfrac{1}{2} (PB + PZ) \text{ sin. } \tfrac{1}{2} (PB - PZ)] - A,$$

whence

$$\text{log. } [\text{cosec. } \tfrac{1}{2} (PB + PZ) \text{ sin. } \tfrac{1}{2} (PB - PZ)] = B + A \qquad (475)$$

and

$$\text{log. tang. } G = A + \text{log. } (-\text{cotan. } \tfrac{1}{2} T) \qquad (476)$$

$$\text{log. tang. } F = A + B + \text{log. } (-\text{cotan. } \tfrac{1}{2} T) \qquad (477)$$

$$= \text{log. tang. } G + B$$

$$\text{log. tang. } \tfrac{1}{2} \text{ alt. non.} = \text{log. cos. } G + \text{log. sec. } F + C. \qquad (478)$$

66. The rule in [B. p. 436] for finding right ascension and declination, when the longitude and latitude are given, may be obtained by a process precisely similar to that for the rule before it.

67. Examples.

1. Calculate the latitude and longitude of the moon, when its right ascension is $4^h\ 42^m\ 56^s$, and its declination $27°\ 21'\ 58''$ N., and the obliquity of the ecliptic $23°\ 27'\ 45''$.

Solution.	$27°\ 21'\ 58''$ N.		tang. 9.71400
$4^h\ 42^m\ 56^s$		tang. 0.45650	cosec. 0.02503
$A = 28°\ 44'\ 12''$ N.		sec. 0.05708	tang. 9.73903
$E = 23°\ 27'\ 45''$ S.			
$B = 5°\ 16'\ 27''$ N.		cos. 9.99816	tang. 8.96524
long. $= 72°\ 53'\ 31''$		tang. 0.51174	sin. 9.98034
lat. $= 5°\ 2'\ 33''$ N.			tang. 8.94558

2. Calculate the values of A, B, and C, for the obliquity $23°\ 77'\ 40''$, and the reduced latitude of $42°\ 12'\ 2''$ N.

Solution. Polar dist. $= 47° \; 47' \; 58''$

$47° \; 47' \; 58''$		
$23° \; 27' \; 40''$		
$\frac{1}{2}$ sum $= 35° \; 37' \; 49''$	sec. 0.09002	tang. $9.85535 = C$
$\frac{1}{2}$ diff. $= 12° \; 10' \; 9''$	cos. 9.99013	tang. 9.33374
	$A = 0.08015$,	$B = 9.47839$

3. Calculate the altitude and longitude of the nonagesimal, when the right ascension of the meridian is $19^h \; 50^m$, the latitude $42° \; 12' \; 2''$ N., and the obliquity $23° \; 27' \; 40''$.

Solution. $T = 19^h \; 50^m + 6^h - 24^h = 1^h \; 50^m$

$\frac{1}{2} \; (1^h \; 50^m)$	cotan. 0.61137	
	$A = 0.08015$	
$G = 101° \; 30' \; 2''$	tang. 0.69152	cos. 9.29968
$90°$	$B = 9.47839$	$C = 9.85535$
$F = 124° \; 4' \; 3''$	tang. 0.16991	sec. 0.25168
long. $= 315° \; 34' \; 5''$	$14° \; 18' \; 40''$	tang. 9.40671
	alt. $= 28° \; 37' \; 20''$.	

4. Calculate the latitude and longitude of the moon, when its right ascension is $18^h \; 27^m \; 12^s$, and its declination $27° \; 49' \; 38''$ S., and the obliquity of the ecliptic $23° \; 27' \; 45''$.

Ans. The ☽'s long. $= 276° \; 1' \; 44''$

Its lat. $= 4° \; 30' \; 27''$ S.

5. Calculate the values of A, B, and C, for Albany in reduced latitude $42° \; 27' \; 39''$, and for the obliquity $23° \; 27' \; 40''$.

Ans. $A = 0.07965$

$B = 9.47565$

$C = 9.85327$

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6. Calculate the longitude and altitude of the nonagesimal, when the obliquity of the ecliptic is $23° 27' 40''$, the latitude $42° 12' 2''$ N., and the R. A. of the meridian $10^h 10^m$.

Ans. The long. $= 138° 30' 25''$
alt. $= 61° 18' 46''$.

7. Calculate the moon's right ascension and declination, when its latitude is $5° 0' 7''$ N., its longitude $64° 54' 1''$, and the obliquity of the ecliptic $23° 27' 45''$.

Ans. Its R. A. $= 4^h 7^m 46^s$.
Its Dec. $= 26° 3' 0''$ N.

68. *Problem. To find the declination of a star.*

Solution. I. Observe its meridian altitude, and its declination is at once found by one of the equations (404–406).

II. If the star does not set, and both its transits are observed, we have

$$p = 90° - \text{Dec.} = \tfrac{1}{2}(A_1 - A'). \qquad (478)$$

69. *Problem. To find the position of the equinoctial points.*

Solution. Since the right ascension of all stars is counted from the vernal equinox, and since the two equinoxes are 12^h apart, the present problem is the same as to find the right ascension of some one of the stars, which may afterwards serve as a fixed point for determining the right ascension of the other stars.

Observe the declination of the sun for several successive noons near the equinox, until two noons are found between which its declination has changed its sign; and observe also the instant of the sun's transit across the meridian on these days, by a clock whose rate of going is known. Then, by supposing the sun's motions in declination and right ascension to be uniform at this time, which they nearly are, the time of the equinox, that is, of the sun's being in the equator, is found by the proportion

> the whole change of declination : either declination $=$ the sideral interval between the transits $- 24^h$: the sideral interval between the transits of the equinox and that of the sun at this declination; . . . . (479)

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and this interval is the difference between the right ascensions of the sun at this declination and the equinox. If the passage of a star had been observed in the same day, the right ascension of the star would have been the interval of sideral time of its passage after that of the vernal equinox.

70. Examples.

1. If the sun's declination is found at one transit to be $7' \, 9''.5$ S., and at the next transit to be $16' \, 31''.1$ N.; what is the sun's right ascension at the second transit, if the sideral interval of the transits is $24^h \, 3^m \, 38^s.21$?

Solution.

$$7' \, 9''.5 + 16' \, 31''.1 = 23' \, 40''.6 = 1420''.6 = \text{ar. co. } 6.84753$$

$$16' \, 31''.1 = 991''.1 \qquad 2.99612$$

$$3^m \, 38^s.21 = 218^s.21 \qquad 2.33887$$

$$\odot\text{'s R. A.} = 0^h \, 2^m \, 32^s.2 \quad 152^s.2 \qquad 2.18252$$

2. If the sun's declination is found at one transit to be $18' \, 38''.8$ S., and at the next transit to be $5' \, 3''.2$ N.; what is the sun's right ascension at the second transit, if the sideral interval of the transits is $24^h \, 3^m \, 38^s.4$?

Ans. $0^h \, 0^m \, 46^s.6$.

3. If the sun's declination is found at one transit to be $5' \, 57''.9$ N., and at the next transit to be $17' \, 26''.3$ S.; what is the sun's right ascension at the second transit, if the sideral interval of the transits is $24^h \, 3^m \, 35^s.71$?

Ans. $12^h \, 2^m \, 40^s.7$.

71. *Problem. To find the obliquity of the ecliptic.*

Solution. Observe the right ascension and declination of the sun, when he is nearly at his greatest declination; that is, when his ascension is nearly 6^h or 18^h. If he were observed at exactly his greatest declination, the observed declination would obviously be the required obliquity. But for any other time, the sun's declination and

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right ascension are the legs of a right triangle, of which the obliquity of the ecliptic is the angle opposite the declination. Hence

$$\text{tang. } \odot\text{'s Dec.} = \text{sin. } \odot\text{'s R. A. tang. obliq.} \qquad (480)$$

Now if we put

$$h = \text{the diff. of } \odot\text{'s R. A. and R. A. of solstice,}$$

we have

$$\cos. h = \frac{\text{tang. } \odot\text{'s Dec.}}{\text{tang. Obliq.}} \qquad (481)$$

and by (346) and (347),

$$\frac{\text{sin. (obliq.} - \odot\text{'s dec.)}}{\text{sin. (obliq.} + \odot\text{'s dec.)}} = \frac{1 - \cos. h}{1 - \cos. h} = \frac{2 \sin.^2 \frac{1}{2} h}{2 \cos.^2 \frac{1}{2} h}$$

$$= \text{tang.}^2 \tfrac{1}{2} h \qquad (482)$$

$$\text{sin. (obliq.} - \odot\text{'s dec.)} = \text{(obliq.} - \odot\text{'s dec.) sin. } 1''$$

$$= \text{tang.}^2 \tfrac{1}{2} h \text{ sin. (obl.} + \odot\text{'s dec.)} \qquad (483)$$

$$\text{obl.} - \odot\text{'s dec.} = \text{cosec. } 1'' \tan.^2 \tfrac{1}{2} h \text{ sin. (obl.} + \odot\text{'s dec.)} \qquad (484)$$

$$= \tfrac{1}{4} h^2 \text{ cosec. } 1'' \tan.^2 1^s \text{ sin. (obl.} + \odot\text{'s dec.)}$$

and the second member of (484) may be regarded as a correction in seconds to be added to the $\odot$'s dec. to obtain the obliquity, and the obliquity in the second member need only be known approximately.

72. Examples.

1. The right ascensions and declinations of the sun on several successive days were as follows:

June 19,	R. A. =	$5^h\ 50^m\ 53^s$,	Dec. =	$23°\ 26'\ 45''.2$ N.
20		5 55 3		23 27 27 .3
21		5 59 12		23 27 44 .7
22		6 3 21		23 27 37 .3
23		6 7 31		23 27 4 .6

To find the obliquity of the ecliptic.

Solution. Assume for the obliquity the greatest observed declina-

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tion, or 23° 27′ 45″, and the corrections of all the observations may be computed in the same way as that of the first, which is thus found,

$\frac{1}{4}$ cosec. $1''$ tang.^{2} $1^s = \frac{225}{4}$ tang. $1''$	6.43570
$h = 9^m\ 7^s = 547^s$	2 log. 5.47598
$23°\ 26'\ 45'' + 23°\ 27'\ 45'' = 46°\ 54'\ 30''$	sin. 9.86348
cor. dec. $= 59'.59$	1.77516
23° 26′ 45″.2	
obliquity $= 23°\ 27'\ 44''.8$	$= 23°\ 27'\ 44''.8$
In the same way the 2d observation gives	23 27 44 .9
the 3d observation gives	23 27 45 .2
the 4th observation gives	23 27 45 .3
the 5th observation gives	23 27 45 .3
sum $=$	117° 18′ 45″.5
The mean $=$	23° 27′ 45″.1

2. The right ascensions and declinations of the sun on several successive days, were as follows:

Dec. 20	☉'s R. A. =	$17^h\ 51^m\ 14^s$	23° 26′ 48″.4 S.
21		17 55 40	23 27 30 .0
22		18 0 7	23 27 44 .0
23		18 4 33	23 27 29 .5
24		18 9 0	23 26 45 .5

what was the obliquity?

Ans. 23° 27′ 44″.7.

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CHAPTER VI.

PRECESSION AND NUTATION.

73. The ecliptic is not a fixed, but a moving plane, and its observed position in the year 1750 has been adopted by astronomers as a *fixed plane*, to which its situation at any other time is referred.

The motion of the ecliptic is shown by the changes in the latitudes of the stars.

74. Celestial motions are generally separated into two portions, *secular* and *periodical.*

Secular motions are those portions of the celestial motions which either remain nearly unchanged, or else are subject to a nearly uniform increase or diminution, which lasts for so many ages, that their limits and times of duration have not yet been determined with any accuracy.

Periodical motions are those whose limits are small, and periods so short, that they have been determined with considerable accuracy.

75. The *true position* of a heavenly body, or of a celestial plane, is that which it actually has; its *mean position* is that which it would have if it were freed from the effects of its periodical motions.

The mean position is, consequently, subject to all the secular changes.

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little less than the half of a second each year, thus causing a diminution of the obliquity of the ecliptic.

Let NAA' (fig. 41) be the fixed plane of 1750, and NA_1 the mean ecliptic for the number of years t after 1750. Let A be the vernal equinox of 1750, and AQ the equator. Let

$$\Pi = NA \text{ and } \pi = \text{the angle } ANA_1\,;$$

then, upon the authority of Bessel, the point of intersection N of the ecliptic, which is called the *node* of the ecliptic, with the fixed plane, has a retrograde motion, by which it approaches A at the annual rate of 5″.18, and if this point could have existed in 1750, its longitude would have been 171° 36′ 10″, so that

$$\Pi = 171^\circ\,36'\,10'' - 5''.18\,t. \tag{485}$$

Moreover, the angle which the mean ecliptic makes with the fixed plane increases at the annual rate of 0″.48892, but this rate of increase is itself decreasing at such a rate, that at the time t this angle is

$$\pi = 0''.48892\,t - 0''.0000030719\,t^2. \tag{486}$$

77. *Problem. To find the change of the mean latitude of a star, which arises from the motion of the ecliptic.*

Solution. Let

$$\begin{aligned} L &= \text{the } \ast\text{'s lat. in 1750} \\ \delta L &= \text{its change of lat.} \\ \varDelta &= \text{its long. in } 1750 - 171^\circ\,36'\,10'' + 5''.18\,t \qquad (487) \\ &= \text{its long. referred to the node of the ecliptic} \\ \delta \varDelta &= \text{its change of long. from the node}\,; \end{aligned}$$

then, if Z (fig. 42) is the pole of the fixed plane, P that of the ecliptic, and B the star; we have

$$PZ = \pi,\ ZB = 90^\circ - L,\ PB = 90^\circ - L - \delta L$$
$$PZB = 90^\circ + \varDelta,\ P = 90^\circ - \varDelta - \delta\varDelta.$$

Draw ZC perpendicular to PB, and we have, since PZ, PC, and CZ are very small,

$$PC = PZ \cos. P = \pi \sin. (\varDelta + \delta\varDelta),$$

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or $$= \pi \sin. A$$

$$\cos. PZ : \cos. PC = \cos. BZ : \cos. BC,$$

or $$BZ = BC$$

$$PC = PB - BZ = -\delta L = \pi \sin. A.$$

$$\delta L = -\pi \sin. A \tag{488}$$

$$= -(0''.48892\, t - 0''.0000030719\, t^2) \sin. A.$$

Again, the triangle ZPB gives, by (354),

$$\sin. \tfrac{1}{2}(PZB + P) : \cos. \tfrac{1}{2}(PZB - P) = \tan. \tfrac{1}{2}\pi : \tan. \tfrac{1}{2}(PB + PZ)$$

But

$$\tfrac{1}{2}(PZB + P) = 90^\circ - \tfrac{1}{2}\delta A,\ \tfrac{1}{2}(PZB - P) = A + \tfrac{1}{2}\delta A,$$

whence $$\delta A = \pi \cos. A \operatorname{tang.} L \tag{489}$$

$$= (0''.48892\, t - 0''.0000030719\, t^2) \cos. A \operatorname{tang.} L.$$

78. The *mean celestial equator* has a motion by which its node upon the fixed plane moves from the node of the ecliptic at the annual rate of about $50''$, while its inclination to the fixed plane has a very small increase proportioned to the square of the time from 1750.

Thus, if AQ (fig. 41) is the equator of 1750, and $A'\,Q'$ that for the time t, so that A is the vernal equinox of 1750, and A_1 that for the time t.

Let $$\psi = AA',\ \omega = NA'Q',$$

then A' moves from A at the annual rate of $50''.340499$, and this rate is diminishing so that at the time t

$$\psi = 50''.340499\, t - 0''.0001217945\, t^2, \tag{490}$$

and the value of ω in the year 1750 was

$$\omega' = 23^\circ\, 28'\, 18'',$$

and is increasing at a rate proportioned to the square of the time, so that

$$\omega = \omega' + 0''.00000984233\, t^2. \tag{491}$$

79. *Problem. To find the change of the mean obliquity of the ecliptic and that of longitude.*

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Solution. Let (fig. 41)

$$NA_1Q' = \omega_1, \quad NA_1 = \psi_1 + \varPi;$$

then, by (369) and (370),

$$\frac{\sin. [\varPi + \frac{1}{2}(\psi + \psi_1)]}{\sin. \frac{1}{2}(\psi - \psi_1)} = \frac{\text{tang.} \frac{1}{2}(\omega + \omega_1)}{\text{tang.} \frac{1}{2}\pi} \tag{492}$$

$$\frac{\cos. [\varPi + \frac{1}{2}(\psi + \psi_1)]}{\cos. \frac{1}{2}(\psi - \psi_1)} = \frac{\text{tang.} \frac{1}{2}(\omega_1 - \omega)}{\text{tang.} \frac{1}{2}\pi}. \tag{493}$$

Now in calculating the parts of $\psi_1 - \psi$ and $\omega_1 - \omega$, which are proportional to the time, we may, since ψ and ψ_1 differ but little as well as ω and ω_1, and since π is small, put

$$\varPi + \tfrac{1}{2}(\psi + \psi_1) = \varPi, \ \sin. \tfrac{1}{2}(\psi - \psi_1) = \tfrac{1}{2}(\psi + \psi_1) \sin. 1''$$

$$\text{tang.} \tfrac{1}{2}\pi = \tfrac{1}{2}\pi \text{ tang. } 1'' = \tfrac{1}{2}\pi \sin. 1'' = \tfrac{1}{2}(0''.48892)\, t \sin. 1''$$

$$\tfrac{1}{2}(\omega + \omega_1) = \omega', \ \text{tang.} \tfrac{1}{2}(\omega_1 - \omega) = \tfrac{1}{2}(\omega_1 - \omega) \sin. 1''$$

$$\cos. \tfrac{1}{2}(\psi - \psi_1) = 1,$$

which, substituted in (492) and (493), give

$$\psi - \psi_1 = 0''.48892\, t \sin. \varPi \text{ cotan. } \omega' \tag{494}$$

$$\omega_1 - \omega = 0''.48892\, t \cos. \varPi, \tag{495}$$

which are thus computed,

0″.48892		9.68924		9.68924
171° 36′ 10″	cos.	9.99532$_n$	sin.	9.16446
— 0″.48368		9.68456$_n$		
		23° 28′ 18″	cotan.	0.36229
0″.164431				9.21599

$$\text{that is, } \omega_1 - \omega = -\,0''.48368\, t \tag{496}$$

$$\psi - \psi_1 = 0''.164431\, t \tag{497}$$

or

$$\omega_1 = 23^\circ\, 28'\, 18'' - 0''.48368\, t \tag{498}$$

$$\varphi_1 = 50''.340499\, t - 0''.164431\, t = 50''.176068\, t. \tag{499}$$

But, in computing the parts of $\omega_1 - \omega$ and $\psi - \psi_1$, which depend upon t^2, we need only retain the part depending upon t^2 in the value of tang. $\frac{1}{2}\pi$, and neglect these parts in the other terms of (492) and (493), we thus have

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$$\sin.\left[\Pi + \tfrac{1}{2}(\psi + \psi_1)\right] = \sin.(\pi + 45''.08\ t) \tag{500}$$

$$= \sin.\ \Pi + 45.''08\ t \sin.\ 1'' \cos.\ \Pi$$

$$\cos.\left[\Pi + \tfrac{1}{2}(\psi + \psi_1)\right] = \cos.(\Pi) - 45''.08\ t \sin.\ 1'' \sin.\ \Pi \tag{501}$$

$$\tan.\tfrac{1}{2}\pi = \tfrac{1}{2}\pi \sin.\ 1'' = \tfrac{1}{2}\sin.\ 1'' (0''.48892\ t - 0''.0000030719\ t^2 \tag{502}$$

$$\text{cotan.}\ \tfrac{1}{2}(\omega + \omega_1) = \text{cotan.}(\omega' - 0''.24184\ t) \tag{503}$$

$$= \frac{1 + 0''.24184\ t \sin.\ 1''\ \text{tang.}\ \omega'}{\text{tang.}\ \omega' - 0''.24184\ t \sin.\ 1''}$$

$$= \text{cotan.}\ \omega' + 0''.24184\ t \sin.\ 1''\ (1 + \text{cotan.}^2\ \omega')$$

$$= \text{cotan.}\ \omega' + 0''.24184\ t \sin.\ 1''\ \text{cosec.}^2\ \omega'$$

$$\cos.\ \tfrac{1}{2}(\psi - \psi_1) = 1,\ \sin.\ \tfrac{1}{2}(\psi - \psi_1) = \tfrac{1}{2}(\psi - \psi_1) \sin.\ 1''$$

$$\sin.\ \tfrac{1}{2}(\omega_1 - \omega) = \tfrac{1}{2}(\omega_1 - \omega) \sin.\ 1'',$$

which, substituted in (492) and (493), give

$$\psi - \psi_1 = 0''.164431\ t + 0''.48892\ t^2 \sin.\ 1'' 45''.08 \cos.\ \Pi\ \text{cotan.}\ \omega'$$

$$+ 0''.48892\ t^2 \sin.\ 1'' \times 0''.24184 \sin.\ \Pi\ \text{cosec.}^2\ \omega' \tag{504}$$

$$- 0''.0000030719\ t^2 \sin.\ \Pi\ \text{cotan.}\ \omega'$$

$$\omega_1 - \omega = - 0''.48368\ t - 0''.48892\ t^2 \sin.\ 1''\ 45''.08 \sin.\ \Pi$$

$$- 0''.0000030719\ t^2 \cos.\ \Pi,$$

which are thus computed,

0''.48892		9.68924		
1''	sin.	4.68557		
45''.08		1.65398		
171° 36' 10''	sin.	9.16446	cos.	9.99532n
— 0''.000015605		5.19325		
+ 0''.000003039		0''.0000030719		4.48741
— 0''.000012566				4.48273n
0''.0000030719		4.48741		

171° 36′ 10″	sin. 9.16446	cos. 9.99532 n	sin. 9.16446
23° 28′ 18″	cotan. 0.36229	0.36229	cosec.² 0.79958
— 0″.000001033	4.01416	sin. 1″ 4.68557	4.68557
	45″.08	1.65398	
	0″.48892	9.68924	9.68924
— 0″.000243445		6.38640 n	
	0″.24184		9.38353
0″.000000528			3.72238
— 0″.000243950			

so that
$$\psi - \psi_1 = 0''.164431\, t - 0''.000243950\, t^2$$
$$\omega_1 - \omega = - 0''.48368\, t - 0''.000012566\, t^2$$
$$\psi_1 = 50.''176068\, t - 0''.0001217945\, t^2 + 0''.000243950\, t^2$$
$$= 50''.176068\, t + 0''.000122156\, t^2 \tag{505}$$
$$\omega_1 = 23^\circ\, 28'\, 18'' - 0''.48368\, t - 0''.000002724\, t^2, \tag{506}$$

or more accurately, from Bessel's *Fundamenta Astronomiæ*,

$$\psi_1 = 50''.176068\, t + 0''.0001221483\, t^2 \tag{507}$$
$$\omega_1 = 23^\circ\, 28'\, 18'' - 0''.48368\, t - 0''.00000272295\, t. \tag{508}$$

These values were afterward changed by Bessel in his *Tabulæ Regiomontanæ* to

$$\psi = 50''.37572\, t - 0''.0001217945\, t^2 \tag{509}$$
$$\psi_1 = 50''.21129\, t + 0''.0001221483\, t^2 \tag{510}$$
$$\psi_1 = 23^\circ\, 28'\, 18'' - 0''.48368\, t - 0''.00000272295\, t^2. \tag{511}$$

But these formulas were obtained from the physical theory, and are, as Bessel says, subject to errors, on account of the uncertainty with regard to some of the data; so that we shall adopt Poisson's formulas, because they agree in the variation of the obliquity almost exactly with Bessel's observations, and shall change the value of ω' to that determined by Bessel from observations; our formulas are, then,

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$$\omega' = 23^\circ\ 28'\ 17''.65 \tag{512}$$

$$\psi = 50''.37572\ t - 0''.00010905\ t^2 \tag{513}$$

$$\psi_1 = 50''.22300\ t + 0''.00011637\ t^2 \tag{514}$$

$$\omega = 23^\circ\ 28'\ 17''.65 + 0''.00008001\ t^2 \tag{515}$$

$$\omega_1 = 23^\circ\ 28'\ 17''.65 - 0''.45692\ t - 0''.000002242\ t^2. \tag{516}$$

If, now, the value of ψ_1 is added to that of $\delta\,A$ (489), the resulting value is the total change of a star's mean longitude.

80. The backward motion ψ_1 of the equinoxes is called the *precession of the equinoxes.*

81. *Problem. To find the intersection of the mean equator with the equator of* 1750 *and its inclination to it.*

Solution. Produce AQ and $A'Q'$ (fig. 41) till they meet at T, and let

$$AT = \Phi,\ A'T = \Phi'$$

and the triangle ATA' gives, by (350, 354, and 369),

$$\cos. \tfrac{1}{2}(\omega' - \omega) : \cos. \tfrac{1}{2}(\omega' + \omega) = \text{tang.} \tfrac{1}{2}\psi : \text{tang.} \tfrac{1}{2}(\Phi' - \Phi) \tag{517}$$

$$\sin. \tfrac{1}{2}(\omega' - \omega) : \sin. \tfrac{1}{2}(\omega' + \omega) = \text{tang.} \tfrac{1}{2}\psi : \text{tang.} \tfrac{1}{2}(\Phi' + \Phi) \tag{518}$$

$$\sin \tfrac{1}{2}(\Phi' + \Phi) : \sin. \tfrac{1}{2}(\Phi' - \Phi) = \text{cotan.} \tfrac{1}{2}T : \cot. \tfrac{1}{2}(\omega' + \omega) \tag{519}$$

so that t^2 may be neglected in all the terms but ψ, and we have

$$1 : \cos. \omega' = \tfrac{1}{2}\psi \sin. 1'' : \tfrac{1}{2}(\Phi' - \Phi) \sin. 1'' \tag{520}$$

$$0 : \sin. \omega' = \tfrac{1}{2}\psi \sin. 1'' : \text{tang.} \tfrac{1}{2}(\Phi' + \Phi) \tag{521}$$

$$1 : \tfrac{1}{2}(\Phi' - \Phi) \sin. 1'' = \text{tang.} \omega' : \tfrac{1}{2} T \sin. 1''. \tag{522}$$

Hence

$$\tfrac{1}{2}(\Phi' + \Phi) = 90^\circ \tag{523}$$

$$\tfrac{1}{2}(\Phi' - \Phi) = \tfrac{1}{2}\psi \cos. \omega' \tag{524}$$

$$T = (\Phi' - \Phi)\ \text{tang.}\ \omega', \tag{525}$$

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which are thus computed,

ω'	cos. 9.96249	cos. 9.96249
$25''.18786$	1.40120	
$23''.103$	1.36369	
$0''.000054525$		5.73660
$0''.000050013$		5.69909
ω'	tang. 9.63771	9.63771
$10''.032$	1.00140	
$0''.000021717$		5.33680

so that

$$\Phi = 90^\circ - 23''.103\, t + 0''.000050013\, t^2 \tag{526}$$

$$T = 20''.0640\, t - 0''.000043434\, t^2. \tag{527}$$

82. *Problem. To find the variation of a star's mean right ascension and declination.*

I. The variation, which arises from the change of the equator's inclination, may be found precisely in the same way in which the variations of latitude and longitude were found in § 77, for a similar change in the position of the ecliptic; so that formulas (488) and (489) give, by substituting for A, L and π,

$$A = \ast\text{'s R. A.} - 90^\circ + 23''.103\, t = R - 90^\circ$$

$$L = \ast\text{'s Dec.} = D,\ \pi = T$$

$$\delta D = -\, T \cos. R \tag{528}$$

$$\delta R = \quad T \sin. R \text{ tang. } D; \tag{529}$$

or instead of counting the value of T and t from 1750, they may be reduced to the beginning of each year, and the square of t may then be neglected.

II. The variation in right ascension is to be increased by the change in the position of the equinox, arising from its precession, which is thus found. Had the ecliptic remained stationary, the equinox would have removed from A to A', so that if AP is perpen-

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dicular to the equator, we should have for the increase of right ascension by (515) and (524),

$$A'P = AA' \cos. AA'P = \psi \cos. \omega \tag{530}$$
$$= (\varphi' - \varphi)$$
$$= 46''.206\, t - 0''.000100026\, t^2.$$

But the equinox advances upon the equator from the motion of the ecliptic by the arc $A'A_1$, which is thus found. We have, by (350),

$$\cos. \tfrac{1}{2} (\omega_1 - \omega) : \cos. \tfrac{1}{2} (\omega_1 + \omega) = \text{tang.} \tfrac{1}{2}\, A'A_1 : \text{tang.} \tfrac{1}{2} (\psi - \psi_1).$$

But $\qquad \cos. \tfrac{1}{2} (\omega_1 - \omega) = 1$

$$\cos. \tfrac{1}{2} (\omega_1 - \omega) = \cos. (\omega' - 0''.22846\, t)$$
$$= \cos. \omega' + 0''.22846\, t \sin. 1'' \sin. \omega'$$
$$\sec. \tfrac{1}{2} (\omega_1 + \omega) = \sec. \omega' - 0''.22846\, t \sin. 1'' \sin. \omega' \sec.^2 \omega'$$
$$\text{tang.} \tfrac{1}{2}\, A'A_1 = \tfrac{1}{2}\, A'A_1 \sin. 1''$$
$$\text{tang.} \tfrac{1}{2} (\psi - \psi_1) = \tfrac{1}{2} (\psi - \psi_1) \sin. 1''$$
$$= \tfrac{1}{2} \sin. 1'' (0''.15272\, t - 0''.00022542\, t^2),$$

whence $A'A_1 = 0''.15272\, t \sec. \omega'$
$$- 0''.00022542\, t^2 \sec. \omega',$$

which is thus computed,

$0''.15272$	9.18390	
ω'	sec. 0.03751	0.03751
$0''.1665$	9.22141	
	$0''.00022542$	6.35299
$0''.00024575$		6.39050

so that

$$A'A_1 = 0''.1665\, t - 0''.00024575\, t^2, \tag{531}$$

and, by (489) and (490),

$$\delta R = 46''.0395\, t + 0''\, 00016593\, t^2 + T \sin. R \text{ tang. } D. \tag{532}$$

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83. By the motions of precession and of diminution of the obliquity, the mean pole of the equator is carried round the pole of the ecliptic, gradually approaching it; but the true pole of the equator has a motion round the mean pole, which is called *nutation*. This motion is in an oval, at the centre of which is the mean pole, and is such that the position of the mean equinox differs from that of the true equinox by the longitude

$$\delta \text{ long.} = i \sin. \Omega + i_1 \sin. 2\,\Omega + i_2 \sin. 2\,☽ + i_3 \sin. 2\,\odot, \quad (533)$$

where

Ω = the mean longitude of that point of intersection of the moon's orbit with the ecliptic, through which the moon ascends from the south to the north side of the ecliptic, and which is called the moon's ascending node,

☽ = the moon's true longitude,

$\odot$ = the sun's true longitude.

The values of i, i_1, i_2, i_3, are given differently by different astronomers, and those which are, at present, adopted in the Nautical Almanac, are

$$\begin{aligned} i &= -17''.2985, & i_1 &= 0''.2082 \\ i_2 &= -0''.2074, & i_3 &= -1''.2550. \end{aligned} \quad (534)$$

This nutation of the pole causes also the true obliquity of the ecliptic to change from the mean obliquity by the quantity

$$\delta\,\omega_1 = k \cos. \Omega + k_1 \cos. 2\,\Omega + k_2 \cos. 2\,☽ + k_3 \cos. 2\,\odot, \quad (535)$$

in which the values of k &c., at present adopted in the Nautical Almanac, are

$$\begin{aligned} k &= 9''.2500, & k_1 &= -00''.0903 \\ k_2 &= 0''.0900, & k_3 &= 0''.5447. \end{aligned} \quad (536)$$

84. *Corollary*. The effect of nutation upon the right ascensions and declinations of the stars may be computed by § 82, and the formulas which are obtained agree with those given in the Nautical Almanac, and which depend upon the terms,

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called C and D in the formulas for Reduction of the Almanac; these terms contain also the changes arising from the mean motion of the equinoxes, and the formulas are so reduced that t is counted from the beginning of each year.

85. Examples.

1. Find the mean obliquity of the ecliptic for the year 1840, and reduce the formulas for finding the variations of right ascension and declination to the beginning of that year.

Solution. In (516) let $t = 1840 - 1750 = 90$,

and it gives

$$\omega_1 = 23° 28' 17''.65 - 41''.12 - 0''.02 = 23° 27' 36''.51.$$

In (527, 528, and 532), let $t = 90 + t$, and neglect the terms depending upon t'^2, so that

$$T = 30' 5''.76 - 0''.35 + 20''.0640\, t' - 0''.0078\, t'$$
$$= 30' 5''.41 + 20''.0562\, t',$$

and the mean variations, counted from the beginning of the year, are

$$\delta' D = 20''.0562\, t' \cos. R$$
$$\delta' R = 46''.0693\, t' + 20''.0562\, t' \sin. R \text{ tang. } D.$$

Finally, the variations arising from nutation are thus found. The change in the obliquity of the ecliptic gives at once, from (488) and (489), by referring the positions to the mean ecliptic instead of to that of 1750,

$$\delta' D = - \delta\, \omega_1 \sin. R$$
$$\delta' R = - \delta\, \omega_1 \cos. R \text{ tang. } D,$$

and the change in the position of the equinox gives by (525, 528, 529, and 530),

$$T = - \delta\, \varDelta \sin. \omega_1$$
$$\delta' D = \delta\, \varDelta \sin. \omega_1 \cos. R$$
$$\delta' R = \delta\, \varDelta \cos. \omega_1 + \delta\, \varDelta \sin. \omega_1 \sin. R \text{ tang. } D.$$

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Hence, if we take

$$46''.0693\ C = 46''.0693\ t' + \delta\ A \cos.\ \omega_1$$

$$c = 46''.0693 + 20''.0562 \sin.\ R \text{ tang. } D$$

$$c' = 20''.0562 \cos.\ R$$

$$d = \cos.\ R \text{ tang. } D$$

$$d' = - \sin.\ R,$$

we have

$$C = t' + \frac{\cos.\ \omega_1}{46''.0693}\ \delta\ A = t' + \frac{\sin.\ \omega_1}{20''.0562}\ \delta\ A$$

$$= t' - 0.3448 \sin.\ ☊ + 0.00415 \sin.\ 2\ ☊$$

$$- 0.00413 \sin.\ 2\ .\ ☽ - 0.02502 \sin.\ 2\ ☉,$$

and the entire changes of declination and right ascension are

$$\delta'\ D = C\ c' - \delta\ \omega\ .\ d'$$

$$\delta'\ R = C\ c - \delta\ \omega\ .\ d,$$

which agree with the formulas in the Nautical Almanac, except in the coefficients of t', which are 46''.0206 and 20''.0426 instead of 46''.0693 and 20''.0620.

If, again, we take

$$f = 46''.0693\ C,$$

$$g \cos.\ G = 20''.0562\ C, \quad g \sin.\ G = - \delta\ \omega,$$

the above formulas become

$$\delta'\ D = g \cos.\ G \cos.\ R - g \sin.\ G \sin.\ R = g \cos.\ (G + R)$$

$$\delta'\ R = f + g \sin.\ R \cos.\ G \text{ tang. } D + g \sin.\ G \cos.\ R \text{ tang. } D$$

$$= f + g \sin.\ (R + G) \text{ tang. } D,$$

as in the Nautical Almanac.

2. Find the annual variations in the right ascension and declination of α Hydræ for the year 1840, and its true place for mean midnight at Greenwich, January 1, 1840; its mean right ascension for January 1, 1839, being $9^h\ 19^m\ 40^s.620$, and its declination $-7°\ 57'\ 49''.50$, and using the numbers of the Nautical Almanac.

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Solution.

$20''.0426$	1.30195	1.30195
$R = 9^h\ 19^m\ 40^s.620$	cos. 9.88374_n	sin. 9.80872
$\delta' D = -15''.335$	1.18569_n	
$D = -7°\ 57'\ 49''.50$		tang. 9.14584_n
$\delta R = 46''.0206 - 1''.8051$		0.25651_n
$= 44''.2155 = 2^s.948.$		

Hence its mean place for Jan. 1, 1840, is

$$R = 9^h\ 19^m\ 43^s.568$$

$$D = -7°\ 58'\ 4''.83.$$

To calculate the effects of nutation, we have

$$☊ = 339°\ 40', \quad ☽ = 242°\ 30', \quad ☉ = 281°\ 15'$$

$$\begin{array}{rlrl} -0.3448\ \text{sin.}\ ☊ = & 0.1205, & 9''.25\ \text{cos.}\ ☊ = & 8''.673 \\ 0.00415\ \text{sin.}\ 2\,☊ = & -0.0027, & -0''.0903\ \text{cos.}\ 2\,☊ = & -0''.068 \\ -0.00413\ \text{sin.}\ 2\,☽ = & -0.0034, & 0''.0900\ \text{cos.}\ 2\,☽ = & -0''.032 \\ -0.02502\ \text{sin.}\ 2\,☉ = & 0.0096, & 0''.5447\ \text{cos.}\ 2\,☉ = & -0''.504 \\ \hline C = t' + & 0.1240, & \delta\omega_1 = & 8''.049 \end{array}$$

$$C\,c' = c'\,t' + 20''.0426 \times 0.1240\ \text{cos.}\ R$$

$$= c'\,t' - 15''.335 \times 0.1240 = c'\,t' - 1''.901$$

$$-\delta\omega\,.\,d' = 8''.049\ \text{sin.}\ R = 5''.181$$

$$C\,c = c\,t' + 0.1240 \times 2^s.948 = c\,t' + 0^s.365$$

$$-\delta\omega\,d = -8''.049\ \text{cos.}\ R\ \text{tang.}\ D = -0''.861 = -0^s.058,$$

whence the variations arising from nutation are

$$\delta' D = 3''.28, \quad \delta' R = 0^s.30,$$

and the true places are

$$D = -7°\ 58'\ 1''.55, \quad R = 9^h\ 19^m\ 43^s.87.$$

3. Find the mean obliquity of the ecliptic for the year 1950, and

reduce the formulas for finding the variations of mean right ascension and declination to the beginning of that year.

Ans. $\omega_1 = 23° 26' 36''.18.$

$$\delta' D = 19''.8903\ t' \cos. R$$

$$\delta' R = 46''.1059\ t' + 19''.8903\ t' \sin. R \text{ tang. } D.$$

4. Find the annual variations in the right ascension and declination of β Ursæ Minoris for the year 1839, and its true place for mean midnight at Greenwich, Aug. 9, 1839; its mean right ascension for Jan. 1, 1839, being $14^h\,51^m\,14^s.943$, its declination $74°\,48'\,48''.89$ N., the longitude of the moon's ascending node for Aug. 9, 1839, being $347°\,17'$, that of the moon $144°\,2'$, and that of the sun $136°\,30'$, and using the constants of the Nautical Almanac, which give for Aug. 9, 1839,

$$f = 32''.33, \quad g = 16''.70, \quad G = 327°\,30'.$$

Ans. Var. in R.A. $= -0^s.277$; var. in Dec. $= 14''.71$;

and for Aug. 9, 1839,

$$R = 14^h\,51^m\,16^s.36$$

$$D = 74°\,48'\,32''.46.$$

5. Calculate the values of f, g, and G for April 1, 1839, mean midnight at Greenwich, when ☊ $= 354°\,10'$, ☉ $= 11°\,34'$, and ☽ is neglected.

Ans. $f = 12''.53, \quad g = 11''.05, \quad G = 299°\,34'.$

In Table XL of the Navigator, the decimal is neglected, and 20 used instead of 20.0562. Table XLIII is calculated from the formulas of Bessel, which differ a little from those of Bailly used in the Nautical Almanac. The construction of these two tables is sufficiently simple from the calculations already given.

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CHAPTER VII.

TIME.

86. The intervals between the successive returns of the mean place of a star to the meridian are precisely equal, and the mean daily motion of the star is perfectly uniform; so that sideral time is adapted to all the wants of astronomy. The instant, which has been adopted as the commencement of the sideral day, is *the upper transit of the vernal equinox.*

The length of the sideral day, which is thus adopted, differs therefore from the true sideral or *star* day by the daily change in the right ascension of the vernal equinox. But this change is annually about 50″ or $3^s.3$, so that the daily change is less than $0^s.01$, and is altogether insensible.

87. *Corollary.* The difference between the sideral time of different places is exactly equal to the difference of the longitude of the places.

88. The interval between two successive upper transits of the sun over the meridian, is called a *solar day;* and the hour angle of the sun is called *solar time.* This is the measure of time best fitted to the common purposes of life.

The intervals between the successive returns of the sun to the meridian, are not exactly equal, but depend upon the variable motion of the sun in right ascension, and can only be determined by an accurate knowledge of this motion.

89. The want of uniformity in the sun's motion in right ascension arises from two different causes.

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I. The sun does not move in the equator, but in the ecliptic.

II. The sun's motion in the ecliptic is not uniform.

The variable motion of the sun along the ecliptic, and its deviations from the plane of the mean ecliptic, cannot be distinctly represented, without reference to the variations of its distance from the earth, and to the nature of the curve which it describes. This portion of the subject, therefore, which involves the determination of the sun's exact daily position, that is, the calculation of its *ephemeris*, must be reserved for the *Physical Astronomy*. It is sufficient, for our present purpose, to know that the sun moves with the greatest velocity when it is nearest the earth, that is, in its *perigee;* and that it moves most slowly when it is farthest from the earth, that is, in its *apogee*.

90. The sun arrives at its perigee about 8 days after the winter solstice, and at its apogee about 8 days after the summer solstice. The mean longitude of the perigee at the beginning of the year 1800, was $279^\circ\ 30'\ 5''$, and it is advancing towards the eastward at the annual rate of about $11''.8$; so that, by adding the precession of the equinoxes, the annual increase of its longitude is about $62''$.

91. To avoid the irregularity of time arising from the want of uniformity of the sun's motion, a fictitious sun, called a *mean sun*, is supposed to move uniformly in the ecliptic, at such a rate as to return to the perigee at the same time with the true sun. A *second mean sun* is also supposed to move in the equator at the same rate with the first mean sun, and to return to each equinox at the same time with the first mean sun.

We shall denote the first mean sun by $\odot_1$, and the second mean sun by $\odot_2$.

92. *Corollary.* The right ascension of the second mean sun is equal to the longitude of the first mean sun.

93. The time which is denoted by the second mean sun is

perfectly uniform in its increase, and is called *mean time;* while that, which is denoted by the true sun, is called *true* or *apparent time;* the difference between mean and true time is called the *equation of time.*

94. The time which it takes the sun to complete a revolution about the earth, is called a *year.*

The time which it takes the mean sun to return to the same longitude, is the *common* or *tropical year.*

The time which it takes it to return to the same star, is the *sideral year;* and the time which it takes it to return to the perigee, is the *anomalistic year.*

The length of the mean tropical year is

$$Y = 365^d\ 5^h\ 48^m\ 47^s.808, \tag{537}$$

so that the daily mean motion of the sun is found by the proportion

$$Y : 1^d = 360° : \text{daily motion} = 59'\ 8''.3302. \tag{538}$$

95. The fraction of a day is necessarily neglected in the length of the year in common life, and the common year is taken equal to 365^d. By this approximation, the error in four years amounts to

$$23^h\ 15^m\ 11^s.232 = 1^d - 44^m\ 48^s.768, \tag{539}$$

or nearly a day, and an additional day is consequently added to the fourth year, which is called the *leap year.* At the end of a century the remaining error amounts to nearly $-0^d.75$, which is noticed by the neglect of three leap years in four centuries. For practical convenience, those years are taken as leap years which are exactly divisible by 4, and the centurial years would thus be leap years, but only those are retained as leap years which are divisible by 400.

96. When the mean sun has returned to the same mean longitude, it has not returned to the same star, because the equinox from which the longitude is counted has retrograded by $50''.223$, so that the mean

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sun has $50''.223$ farther to go, and the time of describing this arc is the fourth term of the proportion

$$59'\ 8''.3302 : 1^d = 50''.223 : 20^m\ 22^s.786, \qquad (540)$$

so that the length of the sideral year is

$$Y_1 = Y + 20^m\ 22^s.786 = 365^d\ 6^h\ 9^m\ 10^s.594. \qquad (541)$$

97. The length of the mean solar day is also different from that of the sideral day, because when the $\odot_2$, in its diurnal motion, returns to the meridian, it is $59'\ 8''.3302$ advanced in right ascension; so that $360°\ 59'\ 8''.3302$ pass the meridian in a solar day, instead of $360°$, which pass in a sideral day. Hence the excess of the solar day above the sideral day, expressed in solar time, is the fourth term of the proportion

$$360°\ 59'\ 8''.3302 : 59'\ 8''.3302 = 1^d : 0^d.0027305$$
$$\text{or } 3^h\ 55^m.9094; \qquad (542)$$

that is, $\quad 1 \text{ sid. day} = 0.9972695 \text{ sol. day},$

or $\quad 24^h \text{ sid. time} = 23^h\ 56^m\ 4^s.0906 \text{ of solar time}; \qquad (543)$

which agrees with (394) and the table for changing sideral to solar time in the Nautical Almanac, and with Table LII of the Navigator.

In the same way this excess expressed in sideral time is the fourth term of the proportion

$$360° : 59'\ 8''.3302 = 1^d : 0^d.002738 \text{ or } 3^m\ 56^s.5554;$$

that is, $\quad 1 \text{ sol. day} = 1.002738 \text{ sid. day}, \qquad (544)$

or $\quad 24^h \text{ sol. time} = 24^h\ 3^m\ 56^s.5554 \text{ sid. time}; \qquad (545)$

which agrees with the table for changing solar to sideral time in the Nautical Almanac, and with Table LI of the Navigator. The remainder of Tables LI and LII, as well as the corresponding ones given in the Nautical Almanac, are calculated by simple proportions from the numbers which are given for 24^h.

The sideral day begins with the transit of the true vernal equinox. At the time of the transit of $\odot_2$, then, that is, at *mean noon*, we have

$$\begin{aligned} \text{the sid. time} &= \text{R. A. of } \odot_2 \text{ from the equinox} \\ &= \text{R. A. of } \odot_2 \text{ from mean equinox} \\ &\qquad + \text{Nutation of equinox in R. A.} \\ &= \text{sun's mean long.} + \text{Nutation in R. A.} \end{aligned} \qquad (546)$$

98. The sun's mean long. for Jan. 1, 1800, at Paris, was found by Bessel to be 279° 54′ 11″.36. Its longitude for Jan. 1, of any other year t, may thus be found. Let f be the remainder after the division of t by 4, the number of days, then, by which Jan. 1 of the year t is removed from Jan. 1, 1800, is

$$\begin{aligned} 365\tfrac{1}{4}\,(t-f)+365\,f &= t\,.\,365\tfrac{1}{4}-\tfrac{1}{4}f \\ &= Y\,.\,t+t\,.\,11^m\,12^s.192-\tfrac{1}{4}f \qquad (547) \\ &= Y\,.\,t+t\,.\,0^d.00778-\tfrac{1}{4}f. \end{aligned}$$

But in Yt days the sun's longitude increases exactly $t\,.\,360°$, which is to be neglected; and its increase in longitude is

$$59'\,8''.3302\,(t+0.00778-\tfrac{1}{4}f)=t\,.27''.61-f\,.\,14'\,47''.083, \qquad (548)$$

or more accurately from Bessel, the mean longitude E, for the first of January of the year $1800+t$ at Paris, is

$$\begin{aligned} E = 279°\ 54'\ 1''.36 + t\ 27''.605844 + t^2\,.\,0''.0001221805 \\ -f\,.\,14'\ 47''.083. \qquad (549) \end{aligned}$$

The mean longitude is found for the first of January, for any other meridian by the following proportion, derived from the interval of time between the $\odot_2$'s passage over this meridian and that of Paris.

$$24^h : \text{long. from Par.} = 59'\ 8''.3302 : \text{change in value of } E. \qquad (550)$$

The sun's mean longitude for any mean noon n of the year after that of the first of Jan. is

$$E + n\,.\,59'\ 8''.3302. \qquad (551)$$

Hence the sideral time of the mean noon n is

$$S = \frac{E}{15} + n\,.\,3^m\ 56^s.555348 + \text{Nutation in R. A.} \qquad (552)$$

so that the solar time of the transit of the equinox from the preceding noon is

$$24^h - S \text{ (converted into solar time).} \qquad (553)$$

99. Examples.

1. Find the sideral interval which corresponds to 10^h of solar time.

Ans. $10^h\ 1^m\ 38^s.5647$.

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2. Find the solar interval which corresponds to 10^h of sideral time.

Ans. $9^h\ 58^m\ 21^s.7044$.

3. Find the sideral interval which corresponds to 10^m of solar time.

Ans. $10^m\ 1^s.6428$.

4. Find the solar interval which corresponds to 10^m of sideral time.

Ans. $9^m\ 58^s.3617$.

5. Find the sideral interval which corresponds to 10^s of solar time.

Ans. $10^s.0274$.

6. Find the solar interval which corresponds to 10^s of sideral time.

Ans. $9^s.9727$.

7. Find the sideral interval which corresponds to $0^s.85$ of solar time.

Ans. $0^s.85233$.

8. Find the solar interval which corresponds to $0^s.85$ of sideral time.

Ans. $0^s.84768$.

9. Find the sun's mean longitude at Greenwich for the mean noon of April 4, 1839, the sideral time at this noon, and the solar time of the transit of the vernal equinox from the preceding noon; the meridian of Greenwich is $9^m\ 21^s.5$ west of that of Paris.

Ans. The sun's mean longitude $= 12°\ 7'\ 3''.02$.

The sideral time of mean noon $= 48^m\ 31^s.27$.

Time of tran. ver. equi. $=$ April 3d, $23^h\ 11^m\ 39^s.68$.

100. *Problem. To find the time by observation.*

Solution. First Method. By equal altitudes.

I. If the star does not change its declination. Observe the times when the star is at equal altitudes before and after passing the

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meridian; the arithmetical mean between these two times is the time of the star's passing the meridian, which compared with the known time of this passage, gives the error of the clock at this time, and the correction of this error gives the time of each observation.

II. When the declination of the star is changing, the time of the star's arriving at the observed altitude A is affected; thus if

$L =$ the latitude,

$D =$ the declination at the meridian,

$\delta D =$ the increase of declination from the meridian,

$h =$ the hour angle, supposing no change in the declination,

$\delta h =$ the increase of the hour angle in time,

we have, by (429),

$$\begin{aligned} \sin. A &= \sin. L \sin. D + \cos. L \cos. D \cos. h \qquad (554) \\ &= \sin. L \sin. (D + \delta D) + \cos. L \cos. (D + \delta D) \cos. (h + \delta h \\ &= \sin. L \sin. D + \delta D \sin. 1'' \sin. L \cos. D + \cos. L \cos. D \cos. h \\ &\quad - \delta D \sin. 1'' \cos. L \sin. D \cos. h - 15\, \delta h \sin. 1'' \cos. L \cos. D \sin. h, \end{aligned}$$

whence

$$\begin{aligned} 0 &= \delta D \sin. L \cos. D - \delta D \cos. L \sin. D \cos. h \\ &\qquad - 15\, \delta h \cos. L \cos. D \sin. h \\ \delta h &= \tfrac{1}{15}\, \delta D \operatorname{tang.} L \operatorname{cosec.} h - \tfrac{1}{15}\, \delta D \operatorname{tang.} D \operatorname{cotan.} h \\ &= \frac{\delta D}{15 \operatorname{cotan.} L \sin. h} - \frac{\delta D}{15 \operatorname{cotan.} D \operatorname{tang.} h}, \qquad (555) \end{aligned}$$

and since the two observations are at nearly the same distance from the meridian, the value of δh is the same for both of them; so that their mean is augmented by δh, and δh is consequently to be subtracted from the mean of the observed times, in order to obtain the true time of the star's passing the meridian.

In calculating the value of δh, its two terms may be calculated separately. Now if $\delta' D$ is the daily variation of the star's declination, we have

$$\delta D = \frac{h\, \delta' D}{24^h} = \frac{2\, h\, \delta' D}{2 \times 24^h}, \qquad (556)$$

and in using proportional logarithms, the proportional logarithm of the hours and minutes of $2\,h$, which is the elapsed time, may be taken as if they were minutes and seconds, provided the same is done with the 24^h in the denominator. Finally, the value of $\delta\,h$ is reduced from minutes and seconds to seconds and thirds by multiplying by 60, so that if M is taken for the denominator of either of the parts of (555), this part P is calculated by the formula

$$\text{Prop. log. } P = -\text{Prop. log.} \frac{2 \times 24^m \times 15}{60} + \text{log. } M + \text{Prop. log. } 2\,h$$
$$+ \text{Prop. log. } \delta'\,D, \qquad (557)$$

which agrees with [B. p. 219], for

$$-\text{Prop. log.} \frac{2 \times 24^m \times 15}{60} = -\text{Prop. log. } 12^m = -1.1761$$
$$= 8.8239. \qquad (558)$$

III. If the altitude at the two observations had differed slightly, the mean time would require to be corrected; for this purpose, let

$\varrho\,A =$ the excess of the second altitude above the first,

$\delta\,h =$ the increase of the hour angle,

and we easily deduce from (554)

$$\cos. A\,\delta\,A = -15 \cos. L \cos. D \sin. h\,\delta\,h, \qquad (559)$$

so that
$$\delta\,h = -\frac{\cos. A\,\delta\,A}{15 \cos. L \cos. D \sin. h}. \qquad (560)$$

The time of the second observation being thus increased by $\delta\,h$, that of the mean is increased by $\frac{1}{2}\,\delta\,h$, which is, therefore the correction to be subtracted from this mean.

The corrections (555) and (560) must be both of them applied when the star is changing its declination, and at the same time the observed altitudes are slightly different.

Second Method. By a single altitude. [B. p. 208-218.]

When a single altitude is observed, there are known in the triangle

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PZB (fig. 35), the three sides, to find the hour angle ZPB, which is thus found by (336),

$$s = \tfrac{1}{2}(z + 90^\circ - L + p) \tag{561}$$

$$\cos.\tfrac{1}{2}h = \sqrt{\left(\frac{\sin.s\sin.(s-z)}{\sin.(90^\circ - L)\sin.p}\right)}, \tag{562}$$

which corresponds to [B. p. 210.]

The hour angle may also be found by (341); thus if we put

$$s' = \tfrac{1}{2}(A + L + p), \tag{563}$$

we have

$$s = \tfrac{1}{2}(180^\circ - A - L + p) = 90^\circ - s' + p = 90^\circ - A - L + s'$$

$$s - p = 90^\circ - s', \quad s - (90^\circ - L) = s' - A,$$

whence

$$\sin.\tfrac{1}{2}h = \sqrt{\left(\frac{\cos.s'\sin.(s'-A)}{\cos.L\sin.p}\right)}, \tag{564}$$

which corresponds to [B. p. 209.]

Third Method. By the distance from a fixed terrestrial object.

If the position of the terrestrial object has been before determined, its hour angle and polar distance may be considered as known.

Hence, if T (fig. 40) is the position of the terrestrial object projected upon the celestial sphere, P the pole, and S the star. Let the distance TS be observed, and let

$$PT = P, \quad PS = p, \quad TS = d,$$

$$TPZ = H, \quad TPS = h', \quad SPZ = h,$$

$$s = \tfrac{1}{2}(P + p + d), \tag{565}$$

we have

$$\sin.\tfrac{1}{2}h' = \sqrt{\left(\frac{\sin.(s-p)\sin.(s-p)}{\sin.P\sin.p}\right)}, \tag{566}$$

or

$$\cos.\tfrac{1}{2}h' = \sqrt{\left(\frac{\sin.s\sin.(s-d)}{\sin.P\sin.p}\right)}, \tag{567}$$

$$h = H + h'.$$

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If the polar distance and hour angle of the terrestrial object is not known, but only its altitude and azimuth, the polar distance and hour angle can be easily found by solving the triangle PZT.

Fourth Method. By a meridian transit. [B. p. 221.]

If the passage of a star is observed over the different wires of a transit instrument, the mean of the observed times is the time of the meridian transit, which should agree with the known time of this transit. This method surpasses all others in accuracy and brevity.

Fifth Method. By a disappearance behind a terrestrial object.

If the instant of a star's disappearance behind a vertical tower has been observed repeatedly with great care, the observed time of this disappearance may afterwards be used for correcting the chronometer. For this purpose, the position of the observer must always be precisely the same. Any change in the right ascension of the star does not affect the star's hour angle, that is, the elapsed time from the meridian transit; this change, consequently, affects the observed time exactly as if the observation were that of a meridian transit.

A small change in the declination of the star affects the hour angle, and therefore the time of observation. Thus, if P (fig. 44) is the pole, Z the zenith, ZSS' the vertical plane of the terrestrial object; then if the polar distance PS is diminished by

$$RS = \delta\, D,$$

the hour angle ZPS is diminished by the angle

$$SPS' = \delta\, h,$$

But $S'R$ is nearly perpendicular to SP, and the sides of $SS'R$ are so small, that their curvature may be neglected, whence

$$RS' = \delta\, D \text{ tang. } S = 15 \cos. D \,.\, \delta\, h,$$

so that
$$\delta\, h = \tfrac{1}{15}\, \delta\, D \text{ tang. } S \sec. D. \tag{568}$$

101. Examples.

1. On May 20, 1823, in latitude 54° 20′ N., the sun was at equal altitudes, the observed interval was $6^h\ 1^m\ 36^s$; find the correction for

the mean of the observed times. The sun's declination is 19° 48′ N., and his daily increase of declination 12′ 44″.

Solution.		8.8239			8.8239
54° 20′	cotan.	9.8559	19° 48′	cotan.	0.4437
$6^h\ 1^m\ 36^s$	sin.	9.8510		tang.	0.0030
$6^m\ 2^s$	P. L.	1.4747			1.4747
12′ 44″	P. L.	1.1503			1.1503
$-\ 12^s.57$		1.1558	$2^s.29$		1.8956
$2^s.29$					

$-\ 10^s.3$ = the required correction.

2. On September 1, 1824, in latitude 46° 50′ N., the interval between the observations, when the sun was at equal altitudes, was $7^h\ 46^m\ 35^s$; the sun's declination was 8° 14′ N., and his daily increase of declination —21′ 49″; what is the correction for the mean of the observations?

Ans. $16^s.4$.

3. On March 5, 1825, in latitude 38° 34′ N., the interval between the observations, when the sun was at equal altitudes, was $8^h\ 29^m\ 28^s$; the sun's declination was 6° 2′ S., and his daily increase of declination was 23′ 9″; what is the correction for the mean of the observations?

Ans. $15^s.4$.

4. On March 27, 1794, in latitude 51° 32′ N., the interval between the observations, when the sun was at equal altitudes, was $7^h\ 29^m\ 55^s$; the sun's declination was 2° 47′ N., and his daily increase of declination 23′ 26″; what is the correction for the mean of the observations?

Ans. $-21^s.7$.

5. In latitude 20° 26′ N., the altitude of Aldebaran, before arriving at the meridian, was found to be 45° 20′, and, after passing the meridian, to be 45° 10′; the interval between the observations was $7^h\ 16^m\ 35^s$, and the declination of Aldebaran was 16° 10′ N.; what is the correction for the mean of the observations?

Ans. 19^s.

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6. In latitude 36° 39′ S., the sun's correct central altitude was found to be 10° 40′, when his declination was 9° 27′ N.; what was the hour angle?*

Ans. $4^h\ 36^m\ 9^s$.

7. In latitude 13° 17′ N., the sun's correct central altitude was found to be 36° 37′, when his declination was 22° 10′ S.; what was the hour angle?

Ans. $2^h\ 42^m\ 52^s$.

8. In latitude 50° 56′ 17″ N., the zenith distance of a terrestrial object was found to be 90° 24′ 28″, and its azimuth 35° 47′ 4″ from the south; what were its polar distance and hour angle?

Ans. Its polar distance = 121° 6′ 43″
Its hour angle = $2^h\ 52^m\ 18^s$.

9. From the preceding terrestrial object, three distances of the sun were found to be 78° 9′ 26″, 77° 39′ 26″, and 77° 29′ 26″, when his declination was 14° 7′ 13″ S.; what were the sun's hour angles, if he was on the opposite side of the meridian from the terrestrial object?

Ans. $2^h\ 45^m\ 49^s$, $2^h\ 43^m\ 26^s$, and $2^h\ 42^m\ 40^s$.

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CHAPTER VIII.

LONGITUDE.

102. *Problem. To find the longitude of a place.*

First Method. By terrestrial measurement.

If the longitude of a place is known, that of another place, which is near it, can be found by measuring the bearing and distance; whence the difference of longitude may be calculated by the rules already given in Navigation.

Second Method. By signals.

The stars, by their diurnal motion, pass round the earth once in 24 sideral hours; hence they arrive at each meridian by a difference of sideral time equal to the difference of longitude. In the same way, the sun passes round the earth once in 24 solar hours; so that it arrives at each meridian by a difference of solar time equal to the difference of longitude. The difference of longitude of two places is, consequently, equal to their difference of time. Now if any signal, as the bursting of a rocket, is observed at two places; the instant of this event, as noticed by the clocks of the two places, gives their difference of time.

Third Method. By a chronometer.

The difference of time of two places can, obviously, be determined by carrying a chronometer, whose rate is well ascertained, from one place to the other; and if the chronometer did not change its rate during the passage, this method would be perfectly accurate.

Fourth Method. By an eclipse of one of Jupiter's satellites.
[B. p. 252.]

The signal of the second method cannot be used, when the places

are more than 20 or 30 miles apart ; and, when the distance is very great, a celestial signal must be used, such as the immersion or emersion of one of Jupiter's satellites. For this purpose, the instant, when any such event would happen to an observer at Greenwich, is inserted in the Nautical Almanac ; and the observer at any other place has only to compare the time of his observation with that of the Almanac to obtain his longitude from Greenwich.

Fifth Method. By an eclipse of the moon. [B. p. 253.]

The beginning or ending of an eclipse of the moon may also be substituted for the signal of the second method to determine the difference of time.

Sixth Method. By a meridian transit of the moon. [B. p. 431.]

The motion of the moon is so rapid, that the instant of its arrival at a given place in the heavens may be used for the signal. Of the elements of its position its right ascension is changing most rapidly, and this element is easily determined at the instant of its passage over the meridian by the difference of time between its passage and that of a known star. The instant of Greenwich time, when the moon's right ascension is equal to the observed right ascension, might be determined from the right ascension, wnich is given in the Nautical Almanac for every hour. But this computation involves the observation of the solar time, whereas the observed interval gives at oncé the sideral time of the observation.

The calculation is then more simple, by means of the Table of Moon-Culminating stars given in the Nautical Almanac, in which the right ascensions of the suitable stars and of the moon's bright limb are given at the instant of their upper transits over the meridian of Greenwich, and also the right ascension of the moon's bright limb at the instant of its lower transit. Hence the difference between the right ascensions of the moon's limb, at two successive transits, is the change of its right ascension in passing from the meridian of Greenwich to that which is 12^h from Greenwich ; so that if the motion in right ascension were perfectly uniform, the right ascension, which corresponded to a given meridian, or the meridian, which corresponded

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to a given right ascension, might be found by the following simple proportion,

$$12^h : \text{long. of place} = \text{diff. of right ascensions for } 12^h : \text{diff. of right ascensions for long. of place}, \quad (569)$$

in which the longitude of the place may be counted from the meridian 12^h from that of Greenwich, provided the change of right ascension for an upper transit is computed from the preceding right ascension, which is that of a lower transit at Greenwich, that is, if the place is in east longitude.

Let then $T =$ long., if west,

or $= 12^h -$ long. (if the long. is east);

and let $A =$ diff. of right ascension for the Greenwich transits, which immediately precede and follow the required or observed transit,

and let $\delta A =$ change of right ascension from the preceding Greenwich transit to the observed transit,

and we have, by (569),

$$12^h : T = A : \delta A, \quad (570)$$

whence

$$\delta A = \frac{A T}{12^h}, \text{ and } T = \frac{12^h \, \delta A}{A}, \quad (571)$$

and if T is reduced to seconds, we have

$$\delta A = \frac{A T}{43200} \quad (572)$$

$$\log. \delta A = \log. A + \log. T + (\text{ar. co.}) \log. 43200$$
$$= \log. A + \log. T + 5.36452 \quad (573)$$

and

$$T = \frac{43200 \, \delta A}{A} \quad (574)$$

$$\log. T = 4.63548 + (\text{ar. co.}) \log. A + \log. \delta A, \quad (575)$$

and formulas (573) and (575) agree with the parts of the rules in the Navigator, which depend upon A, and are independent of the want of uniformity in the moon's motion.

The corrections which arise from the change of the moon's motion may be calculated, on the supposition that this motion is uniformly increasing or decreasing so that the mean motion for any

interval is equal to the motion which it has at the middle instant of that interval. If we put, then,

$$B = \text{the increase of motion in } 12^h, \tag{576}$$

A is not the mean daily motion for the interval of longitude T and the instant $\frac{1}{2}$ T after the meridian transit at Greenwich, but for the interval 12^h and the instant 6^h after this transit. The mean daily motion for the instant $\frac{1}{2}$ T is therefore,

$$A - \frac{(6^h - \frac{1}{2} T) B}{12^h}, \tag{577}$$

so that the correction for A is

$$-\frac{(6^h - \frac{1}{2} T) B}{12^h} = -\frac{(21600^s - \frac{1}{2} T) B}{43200}, \tag{578}$$

and the correction of δA in (572) is

$$\delta B = -\frac{T (21600^s - \frac{1}{2} T)}{(43200^s)^2} B = -\frac{T (43200 - T)}{2 (43200)^2} B, \tag{579}$$

and the value of δB is easily calculated and put into tables, like Table XLV of the Navigator.

In correcting the value of T (574), the correction of δA is to be computed from Table XLV by means of the approximate value of T, and the correction of T is then found by the formula to be

$$\delta T = \frac{43200 \, \delta B}{A}. \tag{580}$$

It only remains, to show how to find the value of B from the Nautical Almanac. Now if A' denotes the motion in right ascension for the 12^h interval of longitude, which precedes that to which A corresponds; and if A'' denotes the motion in right ascension for the 12^h interval of longitude which follows that of A; we have

$$2 B = A'' - A'$$
$$B = \tfrac{1}{2} (A'' - A'), \tag{581}$$

and the calculation agrees entirely with that given in the Navigator.

When the longitude is small, or nearly 12^h, the correction for the variation of motion may be neglected, provided, instead of A, the motion is used which corresponds to the time of the nearest Greenwich transit. Now, in the Nautical Almanac, this motion is given

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for an hour's interval, of which the middle instant is that of the transit, so that if $H =$ this hourly motion, the motion for the time T may be found by the formula

$$1^h : T = H : \delta A$$

whence

$$T = \frac{\delta A \times 1^h}{H} = \frac{2600^s \times \delta A}{H} \tag{582}$$

$$\log. T = 3.55630 + \log. \delta A + (\text{ar. co.}) \log. H, \tag{583}$$

which agrees with [B. p. 432].

The formula (583) may be rendered more correct, if the value of H is taken for the instant $\frac{1}{2}$ T of longitude; and the value can be computed precisely in the same way in which the right ascension was computed for the time T, by noticing the want of uniformity in its increase; and the formula thus corrected is accurate for small differences of longitude.

Seventh Method. By a lunar distance.

The distance of the moon from the sun or a star may be used as the signal; but the true places of these bodies differ from their apparent places, as will be shown in succeeding chapters, so that the observed distance requires to be corrected; and the correction cannot be found without knowing the altitudes of the bodies. It is sufficient, for the present purpose, to know that the difference between the true and apparent places is only a difference of altitude, and not one of azimuth, and that the apparent place of the sun or a star is higher than its true place, while that of the moon is lower. The true distance may, then, be calculated from the observed distance by one of the following methods.

I. Let Z (fig. 45) be the zenith, S the apparent place of the sun or star, and S the true place, M the apparent place of the moon, M' the true place; let

$a =$ the star's apparent alt. $= 90° - ZS$

$a' =$ its true alt. $= 90° - ZS'$

$b =$ the moon's app. alt. $= 90° - ZM$

$b' =$ its true alt. $= 90° - ZM'$

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$$E = \text{the app. dist.} = SM$$
$$E' = \text{the true dist.} = S'M'$$
$$Z = \text{the angle } Z$$
$$\delta\, a = SS' = a - a'$$
$$\delta\, b = MM' = b' - b$$
$$\delta\, b = E' - E.$$

Then the triangles ZSM and $ZS'M'$ give, by (332),

$$2\,(\cos.\tfrac{1}{2}Z)^2 = \frac{\cos.E + \cos.(a+b)}{\cos.a\cos b} = \frac{\cos.E' + \cos.(a'+b')}{\cos.a'\cos.b'}. \quad (584)$$

Let
$$\cos.m = \frac{\cos.a'\cos.b'}{2\cos.a\cos.b}, \quad (585)$$

and we have, by (584),

$$\cos.E' + \cos.(a'+b') = 2\cos.m\cos.E + 2\cos.m\cos.(a+b)$$
$$= \cos.(E+m) + \cos.(E-m) + \cos.(a+b+m) + \cos.(a+b-m)$$
$$\cos.E' = -\cos.(a'+b') + \cos.(E+m) + \cos.(E-m)$$
$$+ \cos.(a+b+m) + \cos.(a+b-m), \quad (586)$$

whence E' can be found by a table of natural sines and cosines, when m has been found from (585).

II. In the same way by (338), we find

$$2\,(\sin.\tfrac{1}{2}Z)^2 = \frac{\cos.(a-b) - \cos.E}{\cos.a\cos.b} = \frac{\cos.(a'-b') - \cos.E'}{\cos.a'\cos.b'} \quad (587)$$

$$\cos.(a'-b') - \cos.E' = 2\cos.m\cos.(a-b) - 2\cos.m\cos.E$$
$$= \cos.(a-b+m) + \cos.(a-b-m) - \cos.(E+m) - \cos.(E-m)$$
$$\cos.E' = \cos.(a'-b') - \cos.(a-b+m) - \cos.(a-b-m)$$
$$+ \cos.(E+m) + \cos.(E-m). \quad (588)$$

III. The correction may be separated into two parts, one of which depends only upon the sun or star, and the other upon the moon; and let

$\delta'\,E$ = the part of $\delta\,E$ which depends upon the sun or star,

$\delta''\,E$ = the part which depends upon the moon.

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Now if the correction were only to be made for the moon, SM would be decreased to SM', whence

$$SM' = E + \delta'' E,$$

and if we put

$$S = ZSM, \ M = ZMS,$$

$$s = \tfrac{1}{2}(a + b + E), \tag{589}$$

the triangles SMM' and SZM give

$$(\sin. \tfrac{1}{2} M)^2 = \frac{\cos. s \sin. (s - a)}{\sin. E \cos. b}$$

$$= \frac{\sin. [E + \frac{1}{2}(\delta'' E - \delta b)] \sin. \frac{1}{2}(\delta'' E + \delta b)}{\sin. \delta b \sin. E}$$

$$= \frac{\delta'' E + \delta b}{2 \delta b} [1 + \tfrac{1}{2} \text{cotan. } E \sin. 1''. (\delta'' E - \delta b)] \tag{590}$$

$$60' + \delta'' E = (59' 42'' - \delta b) + \frac{\cos. s \sin. (s - a)}{\sin. E} \cdot \frac{2 \delta b}{\cos. b}$$

$$+ 18'' - \tfrac{1}{2} \text{cotan. } E \sin. 1'' [(\delta'' E)^2 - (\delta b)^2]. \tag{591}$$

The triangles $SS'M$ and SZM give, by (336) and (340),

$$(\cos. \tfrac{1}{2} S)^2 = \frac{\cos. (s - E) \sin. (s - a)}{\sin. E \cos. a}$$

$$= \frac{\sin. [E + \frac{1}{2}(\delta' E - \delta a)] \sin. \frac{1}{2}(\delta' E + \delta a)}{\sin. \delta a \sin. E}$$

$$= \frac{\delta a + \delta' E}{2 \delta a} \tag{592}$$

$$60' + \delta' E = (60' - \delta a) + \frac{\cos. (s - E) \sin. (s - a)}{\sin. E} \cdot \frac{2 \delta a}{\cos. a}. \tag{593}$$

If now $M'K$ and $S'L$ are drawn perpendicular to MS, and $S'L$ to $M'S$, we have nearly

$$S'M' = E + \delta E = SM' + SL' = E + \delta'' E + SL'$$

$$\delta E = \delta'' E + SL' = \delta'' E + \delta' E + (SL' - \delta' E) \tag{594}$$

$$\delta' E = SL = \delta a \cos. S \tag{595}$$

$$SL' = \delta\, a \cos.\, (S'SL') = \delta\, a \cos.\, (S - MSM')$$

$$= \delta\, a \cos.\, S + \delta\, a \sin.\, S \sin.\, MSM'$$

$$= \delta' E + S'L \sin.\, MSM' \tag{596}$$

$$SL' - \delta' E = S'L \sin.\, MSM'. \tag{597}$$

But from $M'SK$,

$$\sin.\, MSM' = \frac{\sin.\, M'K}{\sin.\, E} = \frac{M'K \sin.\, 1''}{\sin.\, E} \tag{598}$$

whence

$$SL' - \delta' E = \frac{S'L \times M'K \,.\, \sin.\, 1''}{\sin.\, E} \tag{599}$$

and

$$\delta\, E = \delta' E + \delta'' E + \frac{S'L \times M'K \,.\, \sin.\, 1''}{\sin.\, E} \tag{600}$$

$$2° + \delta\, E = (60' + \delta' E) + (60' + \delta'' E) + \frac{S'L \times M'K \sin.\, 1''}{\sin.\, E}, \tag{601}$$

in which 1° is added to $\delta' E$ and $\delta'' E$, in order to render them positive. Now, of $60' + \delta' E$ (593), the part $60' - \delta\, a$ is given in Table XVII or Table XVIII; and the remaining term is computed by proportional logarithms, and is the first correction of the First Method of the Navigator. [B. p. 231.] The proportional logarithm of the factor $2\, \delta\, a \sec.\, a$, is the logarithm of the Table from which $60' - \delta\, a$ is taken.

In the same way, the two first terms of $60' + \delta'' E$ are taken from Table XIX and (591). The remainder of (591) combined with the third term of (601), is computed and inserted in Table XX of the Navigator.

In calculating Table XX, the value of $\delta'' E$ is used, which is obtained from the two first terms of (591); and $S'L$ and $M'K$ are found from $S'SL$ and MKM' in which the sides are so small that their curvature may be neglected, and we have, nearly,

$$S'L = \sqrt{(\delta\, a^2 - \delta' E^2)} \tag{602}$$

$$M'K = \sqrt{(\delta\, b^2 - \delta'' E^2)}. \tag{603}$$

IV. The calculation of the values of $\delta\, a$ and $\delta\, b$ will be fully

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explained in subsequent chapters; but we need only remark, in this place, that the value of δa, for a star, is given in Table XII; for the sun, it is the number of Table XII diminished by that of Table XIV; and for a planet, it is that of Table XII diminished by that of Table X, A. The value of δb is obtained by the formula

$$\delta b = P \cos. b - \delta' b, \tag{604}$$

in which $\delta' b$ is the number of Table XII, and P is the number taken from the Nautical Almanac, and which is called the horizontal parallax. In computing Table XX, the value of P is taken at its mean of 57′ 30″.

In the formulas for the corrections, the zenith distances may be introduced instead of the altitudes, and if we put

$$90^\circ - a = Z,\ 90^\circ - b = z,$$
$$s_1 = \tfrac{1}{2}(z + Z + E), \tag{605}$$

we have, by neglecting the term depending upon the correction of Table XX, as well as the other small quantities,

$$\cos.^2 \tfrac{1}{2} M = \frac{\sin. s_1 \sin. (s_1 - Z)}{\sin. E \sin. z}$$

$$= \frac{\sin. [E + \frac{1}{2}(\delta'' E + \delta b)] \sin. \frac{1}{2}(\delta b - \delta'' E)}{\sin. E \sin. \delta b}$$

$$= \frac{\delta b - \delta'' E}{2 \delta b} \tag{606}$$

$$\delta'' E = \delta b - \frac{2 \sin. s_1 \sin. (s_1 - Z)}{\sin. E \sin. z} \delta b \tag{607}$$

$$\cos.^2 \tfrac{1}{2} S = \frac{\sin. s_1 \sin. (s_1 - z)}{\sin. E \sin. Z} = \frac{\delta' E + \delta a}{2 \delta a}$$

$$\delta' E = -\delta a + \frac{2 \sin. s_1 \sin. (s_1 - z)}{\sin. E \sin. Z} \delta a. \tag{608}$$

Then the second term of the value of $\delta' E$ is the first correction of the Third Method of the Navigator [B. p. 242], and the second term of the value of $\delta'' E$ is the second correction of this method; and the computation from (604, 607, 608) agrees entirely with this method. The third correction is taken from Table XX, as in the first method.

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V. Draw ZN perpendicular to MS, so as to make SN acute. In the right triangle ZSN and ZSM let

$$B = 90° - SN,\ B' = 90° + MN,\ A = \tfrac{1}{2}(B' + B), \tag{609}$$

and we have

$$E = MN + SN = B' - B, \tag{610}$$

and, by Bowditch's Rules for oblique triangles,

$$\cos.\ ZS : \cos.\ ZM = \cos.\ NS : \cos.\ MN,$$

or

$$\sin.\ a : \sin.\ b = \sin.\ B : \sin.\ B'; \tag{611}$$

and, by the theory of proportions,

$$\frac{\sin.\ a + \sin.\ b}{\sin.\ b - \sin.\ a} = \frac{\sin.\ B + \sin.\ B'}{\sin.\ B' - \sin.\ B},$$

that is,

$$\frac{\text{tang.}\ \frac{1}{2}(a + b)}{\text{tang.}\ \frac{1}{2}(b - a)} = \frac{\text{tang.}\ A}{\text{tang.}\ \frac{1}{2} E} \tag{612}$$

$$\text{tang.}\ A = \text{tang.}\ \tfrac{1}{2}(a + b)\ \text{cotan.}\ \tfrac{1}{2}(b - a)\ \text{tang.}\ \tfrac{1}{2} E \tag{613}$$

$$B' = A + \tfrac{1}{2} E,\ B = A - \tfrac{1}{2} E, \tag{614}$$

and the right triangles ZSN, MZN, SLS', MKM', give

$$\cos.\ S = \frac{\delta' E}{\delta a} = \text{cotan.}\ ZS\ \text{tang.}\ a\ \text{cotan.}\ B$$

$$-\cos.\ M = \frac{\delta'' E}{\delta b} = -\ \text{cot.}\ ZM\ \text{tang.}\ MN = \text{tang.}\ b\ \text{cotan.}\ B'$$

$$\delta' E = \delta a\ \text{tang.}\ a\ \text{cotan.}\ B \tag{615}$$

$$\delta'' E = \delta b\ \text{tang.}\ b\ \text{cotan.}\ B', \tag{616}$$

and the formulas (613–616) correspond to the Fourth Method of the Navigator. [B. p. 243.]

It may be observed, that since cotan $\frac{1}{2}(b - a)$ is the only term of (613) which can change its sign, A is acute when b is greater than a, and obtuse when b is less than a.

VI. The most important of corrections of the distance arise from that term of δb (604), which depends upon the parallax. If we consider this, therefore, as the only correction of the moon's alti-

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tude, we may calculate the corrections of the distance arising from it by putting

$$\delta\, b = MM' = P \cos.\, b. \tag{617}$$

The triangles ZSM and $M'MK$, give then

$$\cos.\, M = -\frac{\delta''\, E}{P \cos.\, b} = \frac{\sin.\, a - \cos.\, E \sin.\, b}{\sin.\, E \cos.\, b} \tag{618}$$

$$\delta''\, E = -P \sin.\, a \operatorname{cosec} E + P \operatorname{cotan.} E \sin.\, b, \tag{619}$$

aand if we put

$$\delta_1\, E = P \sin.\, a \operatorname{cosec.} E \tag{620}$$

$$\delta_2\, E = \pm P \operatorname{cotan.} E \sin.\, b, \tag{621}$$

in which the signs are taken so that $\delta_2\, E$ is always positive, we have

$$\delta''\, E = -\delta_1\, E \pm \delta_2\, E$$

$$10^\circ + \delta''\, E = (5^\circ - \delta_1\, E) + (5^\circ \pm \delta_2\, E). \tag{623}$$

Now Table XLVII is a common table of proportional logarithms, like Table XXII; but the angle which is placed at the top of the table is

$$5^\circ - \text{the angle of Table XXII}, \tag{624}$$

and the angle at the bottom of the table is

$$5^\circ + \text{the angle of Table XXII}; \tag{625}$$

so that the terms of (623) may be directly obtained from these tables; and this method of computing the corrections, which depend upon the moon's parallax, agrees with the second method of the Navigator. [B. p. 239.]

The remaining corrections may be computed from the formulas (607 and 608), and the corrections of Table XX may be neglected, provided the value of E is corrected for the parallax. These combined corrections may be inserted in a table like Table XLVIII, which serves for the star, and, by means of the part P, for the sun; or like Tables XLIX and L, which serve for the planets. In calculating those tables, the moon's horizontal parallax is taken at its mean value of $57'\ 30''$; and the planet's or sun's parallax in altitude is obtained from the formula

$$\delta'\, a = -P \cos.\, a,$$

in which P is the horizontal parallax. The value of P, used in the construction of the part P of Table XLVIII, is $8''.6$; that used for Table XLIX is $35''$; and since these corrections are proportional to the parallax, they are easily reduced to any other parallax. This reduction is actually made in Table L.

VII. The value of $\delta'' E$ (618), might be found by the formula

$$\delta'' E = -\frac{2 \sin. a - \sin. (b + E) - \sin. (b - E)}{2 \sin. E} P, \qquad (626)$$

which is easily calculated by means of the table of natural sines and cosines.

VIII. The true distance may be obtained from observation by either of the preceding methods, and the time of the observation must be compared with the time when the distance is the same to an observer at Greenwich. Now this latter time can be obtained from the Nautical Almanac by precisely the same process of interpolation, which has been applied to the changes of right ascension. The distances are given in the Nautical Almanac for every three hours, and the proportional logarithm of the difference of these distances. If, then, the distance increases uniformly at the rate of increase, F, for every three hours; the interval T, at which it has increased by the quantity F', is found by the proportion

$$F : F' = 3^h : T \qquad (627)$$

$$\text{Prop. log. } T = \text{Prop. log. } F' - \text{Prop. log. } F + \text{Prop. log. } 3^h. \qquad (628)$$

But

$$\text{Prop. log. } 3^h = 0; \qquad (629)$$

and if we put

$$\text{Prop. log. } F = Q, \qquad (630)$$

(628) becomes

$$\text{Prop. log. } T = \text{Prop. log. } F' - Q. \qquad (631)$$

If the distance increased uniformly, the value of Q would be invariable; but Q is variable, and must be regarded as belonging to the middle instant of the interval to which it belongs; and it increases while the distance decreases, and the reverse. Let then

$\delta\, Q =$ the decrease of Q in three hours,

$\delta\, T =$ the correction of T, arising from the change of Q,

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and the value of Q for the interval T is

$$Q + \frac{90^m - \frac{1}{2}T}{180^m}\,\delta\,Q = Q + \delta'\,Q, \tag{632}$$

so that by (631) and (399)

$$\text{Prop. log. } (T + \delta\,T) = \text{Prop. log. } T - \delta'\,Q \tag{633}$$

$$\log.\ (T + \delta\,T) = \log.\ T + \delta'\,Q \tag{634}$$

$$\log.\ (T + \delta\,T) - \log.\ T = \log.\left(1 + \frac{\delta\,T}{T}\right) = \delta'\,Q. \tag{635}$$

But if in (167) we substitute

$$\frac{\delta\,T}{T} = i \tag{636}$$

we have, by (635),

$$\log.\ e = \left(1 + \frac{\delta\,T}{T}\right) = \frac{T\,\delta'\,Q}{\delta\,T}, \tag{637}$$

so that by (632) and (164)

$$\delta\,T = \frac{T\,\delta'\,Q}{\log.\ e} = \frac{(180^m - T)\,T\,\delta\,Q}{2 \times 180^m. \times 0.434} \tag{638}$$

$$= \frac{(180^m - T)\,T\,\delta\,Q}{156^m}; \tag{639}$$

and the table [B. p. 245] for correcting by second differences may be calculated by this formula; and, in order to obtain the value of $\delta\,T$ expressed in seconds, the factor T should be expressed in seconds, while $(180^m - T)$ is expressed in minutes; and it must not be forgotten, that the proportional logarithms are decimals.

IX. When the distance is observed for a star, whose distance is not given in the Nautical Almanac, the Greenwich time of the observation can be found approximately by adding the assumed longitude, if west, to the observed time, or subtracting it if east; or the time can be taken from the chronometer if it is regulated to Greenwich time.

Find, in the Nautical Almanac, the right ascension and declination of the star, and the declination of the moon, for this time. Then, if T and S (fig. 40) are supposed to be the moon and star, and P the

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pole of the equator, D and D' their declinations, disregarding their names, so that their polar distances are $90° \pm D$ and $90° \pm D'$, and if R' is their difference of right ascensions, we have, when their declinations are of the same name, by putting

$$S = \tfrac{1}{2}(D + D' + E) \tag{640}$$

$$\cos. \tfrac{1}{2} R' = \cos. \tfrac{1}{2} SPT = \sqrt{\left(\frac{\cos. S \cos. (S - E)}{\cos. D \cos. D'}\right)}. \tag{641}$$

But if the declinations are of the same name,

$$\sin. \tfrac{1}{2} R' \sin. \tfrac{1}{2} SPT = \sqrt{\left(\frac{\sin. S \sin. (S - E)}{\cos. D \cos. D'}\right)}, \tag{642}$$

and the right ascension of the moon being thus found, the Greenwich time, when it has this right ascension, is easily found from the moon's hourly ephemeris in the Nautical Almanac, and this method is the same with that in [B. p. 428].

X. The latitudes and longitudes may be used instead of the right ascensions and declinations, and the calculation will be as in [B. p. 427]. The variation of daily motion is, in this case, to be had regard to, precisely as explained in (606–611).

XI. The distances of the Nautical Almanac can be calculated from the right ascensions and declinations of the sun, moon, and stars, or their latitudes and longitudes, by resolving the triangles TPS (fig. 40) by either of the methods which have been given, when two sides and the included angle are known, as in [B. p. 434].

In calculating the distance of the sun and moon, the latitude of the sun may be usually neglected; so that if SR (fig. 46) is an arc of the ecliptic, S the sun's place, M the moon's, and MR perpendicular to SR,

$$MR = L = \text{the moon's latitude,}$$

$$SR = L_1 = \text{the diff. of long. of ☉ and ☽,}$$

and

$$\cos. E = \cos. SM = \cos. L \cos. L_1, \tag{643}$$

as in [B. p. 433].

It would, however, be rather more accurate to take

$$L = \text{the diff. of lat. of ☉ and ☽.}$$

XII. The determination of the longitude by solar eclipses and occultations, will be reserved for another chapter.

103. Examples.

1. Calculate the correction of Table XLV, when

$$T = 1^h\ 50^m,\ \text{and}\ B = 9^m = 540^s.$$

Solution.	$1^h\ 50^m$ P. L. $1^m\ 50^s$	ar. co.	8.0080
$12^h - 1^h\ 50^m = 10^h\ 10^m$	P. L. $10^m\ 10^s$	ar. co.	8.7519
	2 P. L. 12^m		2.3522
$\frac{1}{2} B = 270^s$			2.4314
corr. $= 34^s.9$			1.5435

2. Calculate the correction of Table XLV, when

$$T = 3^h\ 10^m,\ \text{and}\ B = 11^m.$$

Ans. $64^s.1$.

3. Find the right ascension of the moon's bright limb, Sept. 25, 1830, at the time of the transit over the meridian of New York. The right ascension of the moon for the two preceding and the two following transits at Greenwich are

Sept. 25.	Moon II. L. T.	$2^h\ 0^m\ 36^s.69$
	Moon II. U. T.	2 30 38 .08
Sept. 26.	Moon II. L. T.	3 1 33 .18
	Moon II. U. T.	3 33 19 .89
The Longitude of New York is		4 56 4 .5

Ans. $2^h\ 43^m\ 14^s.4$.

4. At a place in west longitude, Oct. 25, 1839, the moon's bright limb passed the meridian $10^m\ 6^s.83$ sidereal time, before the star C. Tauri; find the longitude of the place of observation.

The right ascension of the star C. Tauri was $5^h\ 43^m\ 16^s.84$, and those of the moon

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Oct. 25.	Moon II. L. T. $4^h\ 43^m\ 53^s.55$
	Moon II. U. T. 5 18 28 .40
Oct. 26.	Moon II. L. T. 5 52 51 .91
	Moon II. U. T. 6 26 40 .00

Ans. 76° 53′ 33″ W.

5. Find the moon's parallax in altitude, and the correction and logarithm of Table XIX, when the altitude is 40° 40′, and the horizontal parallax is 58′.

Solution.	58′	P. L. 0.4918
	40° 40′	sec. 0.1200
Parallax in alt. =	44′	P. L. 06118
By Table XII. Refrac. =	1′ 6″	9.6990
Corr. = 16′ 48″ * = 59′ 42″ —	42′ 54″	P. L. 0.6228
	Log. of Table XIX =	0.2018

6. Find the correction and logarithm of Table XVII for a star, when the altitude is 13° 15′.

Ans. Corr. = 56′ 2″, Log. = 1.3433.

7. Find the correction and logarithm of Table XVII for Venus or Mars, when the parallax is 20″, and the altitude 24° 30′.

Ans. Corr. = 58′ 14″, Log. = 1.6647.

8. Find the correction and logarithm of Table XVIII, when the altitude is 56°.

Ans. Corr. = 59′ 26″, Log. = 1.9544.

9. Find the correction and logarithm of Table XIX, when the altitude is 70°, and the horizontal parallax 54′.

Ans. Corr. = 41′ 34″, Log. = 0.2299.

* The numbers of Table XIX are so disposed in the Navigator, that the corrections of proportional parts of parallax are all additive. This is effected by placing each number opposite that parallax, which is 10″ less than the one to which it belongs. There is, therefore, a correction for 0″ of parallax.

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10. Compute the value of the auxiliary angle m, in the first and second methods of correcting the lunar distance, when the moon's apparent altitude is $40° 40'$, its horizontal parallax $58'$, and the sun's apparent altitude $70°$.

Solution. The values of m might be computed directly from (585), but it is more convenient to obtain it by some process of approximation. For this purpose let

$$m = 60° + \delta m,$$

and we have

$$2 \cos. (60° + \delta m) = \frac{\cos. (b + \delta b)}{\cos. b} \frac{\cos. (a - \delta a)}{\cos. a}$$

$$= 2 \cos. 60° \cos. \delta m - 2 \sin. 60° \sin. \delta m \qquad (644)$$

$$= (\cos. \delta b - \text{tang.}\, b \sin. \delta b) (\cos. \delta a + \text{tang.}\, a \sin. \delta a),$$

in which we may put

$$2 \cos. 60° = 1, \ \cos. \delta b = 1 - 2 \sin.^2 \tfrac{1}{2} \delta b = 1 - \tfrac{1}{2} \delta b^2 \sin.^2 1''$$

$$\cos. \delta m = 1 - \tfrac{1}{2} \delta m^2 \sin.^2 1'',$$

and (644) becomes

$$2 \delta m \sin. 60° = \delta b \text{ tang. } b. - \delta a \text{ tang } a \qquad (645)$$

$$+ \tfrac{1}{2} (\delta b^2 - \delta m^2) \sin. 1''.$$

But if we take

$$e = 2 \delta b \sec. b . \text{and } e = 2 \delta a \sec. a,$$

Prop. log. e is the logarithm of Table XIX, and Prop. log. e' is the corresponding logarithm for the sun, star, or planet; and by (645),

$$\delta m = \tfrac{1}{4} e \sin. b . \text{cosec. } 60° - \tfrac{1}{4} e' \sin. a \text{ cosec. } 60° \qquad (646)$$

$$+ \tfrac{1}{2} (\delta b^2 - \delta m^2) \sin. 1'' \text{ cotan. } 60°,$$

whence in the present case

e	P. L.	0.2018	e'	P. L.	2.0173
40° 40′	cosec.	0.1860	70°	cosec.	0.0270
60°	sin.	9.9375			9.9375
1° 25′ 7″		0.3253	1′ 53″		1.9818

approx. $\delta m = \frac{1}{4}(1^\circ 25' 7'' - 1' 53'') = \frac{1}{4}(1^\circ 23' 14'') = 20' 48'' = 1248''$

$\delta b = 42' 54'' = 2574''$		
$\delta b + \delta m = 3822$		3.5823
$\delta b - \delta m = 1326$		3.1225
$1''$	sin.	4.6856
60°	cotan.	9.7614
corr. $\delta m = 7'' = \frac{1}{2}(14'')$		1.1518
$\delta m = 20' 48'' + 7'' = 20' 55''$.		

11. Compute the value of the auxiliary angle m, when the moon's apparent altitude is $25^\circ 30'$, the horizontal parallax $60'$, and the star's apparent altitude 10°.

Ans. $60^\circ 14' 3''$.

12. Find the correction of Table XX, when the distance is 25°, the sun's altitude 10°, and the moon's altitude 25°.

Solution. We should find, in this case,

$\delta b =$	$50' 6''$		$\delta a =$	$5' 6''$	
$\delta'' E =$	$-27' 22''$		$\delta' E =$	$-3' 15''$	
$\delta b - \delta'' E =$	$1^\circ 17' 28'' = 4648''$		$\delta a - \delta' E =$	$8' 21'' = 501''$	
$\delta b + \delta'' E =$	$22' 44''$		$\delta a + \delta' E =$	$1' 51'' = 111''$	
$22' 44'' =$	P. L.	0.8986			0.899
$1^\circ 17' 28'' = 4648''$	(ar. co.)	6.3327		P. L.	0.366
25°	tang.	9.6687		2 sin.	9.252
$1''$	cosec.	5.3144	$1''$	2 cosec.	0.629
$1' 6'' = 66''$		2.2144	$501''$	(ar. co.)	7.300
$\frac{1}{2}(66'') = 33''$			$111''$	(ar. co.)	7.955
					2)6.401
$24'' =$			$18'' + 6''$		3.200
$57'' =$ corr. Table XX.					

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13. Calculate the correction of Table XX, when the distance is 120°, the sun's altitude 20°, and the moon's altitude 10°.

Ans. 10″.

14. Calculate the corrections of Tables XLVIII, XLIX, and L, when the apparent distance is 28°, the moon's apparent altitude 38°, the planet's apparent altitude 18°, and its horizontal parallax 16″.

Solution.

57′ 30″	P. L.	0.4956		0.4956
18°	cosec.	0.5100	38° cosec.	0.2107
28°	sin.	9.6716	tang.	9.7257
5° — 1st cor. = 4° 22′ 9″		0.6772	5° + 2d cor. = 6° 6′ 34″	0.4320
6° 6′ 34″			moon's par. in alt. = 45′	
28°			moon's approx. alt. = 38° 45′	
28° 29′ = approx. dist.				
18°			45′ + 29′ = 74′ = 4440″ ar. co.	6.3526
38° 45′			45′ — 29′ = 16′ P. L.	1.0512
28° 22′ = ½ sum	tang.	9.73235	28° tang.	9.7257
10° 22′ = ½ diff.	cotan.	0.73771	1″ cosec.	5.3144
½ (28°) = 14°	tang.	9.39677	2)39″	2.4439
A = 36° 21′	tang.	9.86683	20″	
1st tang. = 22° 21′	tang.	9.6140		9.614
18°	cotan.	0.4882		0.488
By Table XII 2′ 54″	P. L.	1.7929	Table X, A. 33″ P. L.	2.515
2′ 18″		1.8951	25″ = cor. Table XIX	2.617
2d ang. = 50° 21′	tang.	0.0816	$\frac{16}{35}$ × 25″ = 11″ = cor. Table L.	
38°	cotan.	0.1072		
By Table XII 1′ 13″	P. L.	2.1701		
47″		2.3589		

Cor. Table XLVIII = 2′ 18″ — 47″ + 20″ = 1′ 51″.

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15. Calculate the corrections of Tables XLVIII, XLIX, and L, when the apparent distance is 60°, the moon's apparent altitude 50°, the planet's apparent altitude 30°, and its horizontal parallax 30″.

Ans. Cor. Table XLVIII $= 1' \, 25''$
XLIX $= -21''$
L $= -18''$.

16. Find the correction of the Table [B. p. 245] for the interval of $2^h \, 30^m$, and the difference of the Proportional Logarithms equal to 88.

Ans. 15^s

17. If the observed distance were 45° 34′ 10″, the moon's apparent altitude 22° 19′, its horizontal parallax 60′ 19″, the planet's apparent altitude 42° 12′, its horizontal parallax 15″.3; what is the true distance?

Solution. I. In this case $m = 60° \, 12' \, 28''$

$$a = 42° \, 12' \qquad \delta a = 51'' \qquad a' = 42° \, 11' \, 9''$$
$$b = 22° \, 19' \qquad \delta b = 53' \, 31'' \qquad b' = 23° \, 12' \, 31''$$
$$a' + b' = 65° \, 23' \, 40'' \quad -\text{N. cos.} = -0.41637 \qquad E = 45° \, 34' \, 10''$$
$$E + m = 105° \, 46' \, 38'' \quad \text{N. cos.} = -0.27189$$
$$a + b + m = 124° \, 43' \, 28'' \quad \text{N. cos.} = -0.56963$$
$$\overline{-1.25789}$$
$$E - m = -14° \, 38' \, 18'' \quad \text{N. cos.} = 0.96754$$
$$a + b - m = 4° \, 18' \, 32'' \quad \text{N. cos.} = 0.99717$$
$$E' = 45° \, 1' \, 24'' \quad \text{N. cos.} = 0.70682$$

II.
$$a - b + m = 80° \, 5' \, 28'' \quad -\text{N. cos.} = -0.17208$$
$$a - b - m = -40 \; 19 \; 28 \quad -\text{N. cos.} = -0.76239$$
$$E + m = 105 \; 46 \; 38 \quad \text{N. cos.} = -0.27189$$
$$\overline{-1.20636}$$
$$a' - b' = 18° \, 58' \, 38'' \quad \text{N. cos.} = 0.94565$$
$$E - m = -14 \; 38 \; 18 \quad \text{N. cos.} = 0.96754$$
$$\overline{}$$
$$E' = 45 \; 1 \; 21 \quad \text{N. cos.} = 0.70683$$

III. $s = \frac{1}{2}(a + b + E) = 55° 2' 35''$			sec. 0.2419
$E = 45° 34' 10''$	sin. 9.8538		9.8538
$s - a = 12\ 50\ 35$	cosec. 0.6531		0.6531
$s - E = 9\ 28\ 25$	sec. 0.0060	$6' 11''$. T. XIX.	0.1920
$59' 8''$. Table XVII	1.8907	$20' 38''$	P. L. 0.9408
$43''$.	P. L. 2.4036	$32''$	Table XX.
$59°.51$		$27' 21''$	

$$E' = 45° 34' 10'' + 59' 51'' + 27' 21'' - 2° = 45° 1' 22''.$$

IV.	$Z = 47° 48'$	$z = 67° 41'$	
$s_1 = 80° 31' 35''$	cosec. 0.0060		
$E = 45° 34' 10''$	sin. 9.8538		
	9.6990		
	9.5588		9.5588
$Z = 47° 48'$	sin. 9.8697	$z = 67° 41'$	sin. 9.9662
$s_1 - z = 12° 50' 35''$	cosec. 0.6531		
$s_1 - Z = 32° 43' 35''$			cosec. 0.2671
$\delta a = 51''$	P. L. 2.3259	$\delta b = 53' 31''$	P. L. 0.5268
1st cor. $= 42''$	P. L. 2.4075	2d cor. $1° 26' 22''$	0.3189
$\delta b = 53' 31''$		$\delta a = 51''$	

$$E' = 54' 13'' + 45° 34' 10'' + 31'' - 18'' - 1° 27' 13'' = 45° 1' 23''.$$

V.	$\frac{1}{2}(a + b) = 32° 15' 30''$	tang. 9.80014
	$\frac{1}{2}(a - b) = 9\ 56\ 30$	cotan. 0.75627
	$\frac{1}{2} E = 22\ 47\ 5$	tang. 9.62330
	$A = 123\ 28\ 14$	tang. 0.17971

1st ang. $= 100° 41' 9''$	tang. 0.7242		
2d ang. $= 146° 15' 19''$			tang. 9.8248
$a = 42° 12'$	cotan. 0.0425	$b = 22° 19'$	cotan. 0.3867
$\delta a =$	$51''$ P. L. 2.3259	$\delta b = 53' 31''$	P. L. 0.5268
1st cor. $=$	$-9''$ P. L. 3.0926		
2d cor. $= -32' 53''$			P. L. 0.7383

$$E' = 45° 34' 10'' - 9'' - 32' 53'' + 31'' - 18'' = 45° 1' 21''.$$

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VI.	$60' \, 19''$ P. L. 0.4748			0.4748
$a = 42^\circ \, 12'$	cosec. 0.1728	$b = 22^\circ \, 19'$		cosec. 0.4205
$E = 45^\circ \, 34' \, 10''$	sin. 9.8538			tang. 0.0086
1st cor. $= 4^\circ \; 3' \, 16''$	0.5014	2d cor. $= 5^\circ \, 22' \, 28''$		0.9039

Cor. Table XLVIII, XLIX, and L $= 1' \, 31''$

$$E' = 45^\circ \, 34' \, 10'' + 4^\circ \, 3' \, 16'' + 5^\circ \, 22' \, 28'' + 1' \, 31'' - 10^\circ = 45^\circ \, 1' \, 25''.$$

VII.

$a =$	$42^\circ \, 12'$	N. sin.	0.67172		
$b + E =$	$67^\circ \, 53' \, 10''$,	$\frac{1}{2}$ N. sin.	$-$ 0.46322	$60' \, 19''$ P. L.	0.4748
$b - E =$	$-23^\circ \, 15' \, 10''$,	$\frac{1}{2}$ N. sin.	0.19739		
			0.40589	ar. co.	0.3916
$E =$	$45^\circ \, 34' \, 10''$,			sin.	9.8538
Cor. Table XLVIII, &c. $=$	$1' \, 31''$		cor. $= - 34' \, 17''$		0.7202

$$E' = 45^\circ \, 34' \, 10'' + 1' \, 31'' - 34' \, 17'' = 45^\circ \, 1' \, 24''.$$

18. The apparent distance of the sun and moon is $70^\circ \, 50' \, 33''$; the moon's apparent altitude is $35^\circ \, 45' \, 4''$, its horizontal parallax is $54' \, 24''$; the sun's apparent altitude is $70^\circ \, 48' \, 1''$; what is the true distance?

In this example $m \; 60^\circ \, 17' \, 28''$.

Ans. $70^\circ \, 8' \, 47''$.

19. The apparent distance of a star from the moon is $31^\circ \, 13' \, 26''$; the moon's apparent altitude is $8^\circ \, 26' \, 13''$, its horizontal parallax is $60'$, the star's apparent altitude is $35^\circ \, 40'$; what is the true distance?

In this example $m \; 60^\circ \, 4' \, 16''$.

Ans. $30^\circ \, 24' \, 48''$.

20. Find the Greenwich time, Oct. 3, 1839, when the moon's distance from the sun was $38^\circ \, 12' \, 9''$.

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Solution.

Distance 1839, Oct. 3, 15^h	$38° 59' 21''$	P. L. 0.3180
	$38\ 12\ 9$	
18^h P. L. 3189	$47' 12''$	P. L. 0.5813
3180	$T = 1^h 33^m 16^s$	P. L. 0.2633
9 cor. T. =	-3^s	

Greenwich time $= 16^h 38^m 7^s$.

21. Find the Greenwich time, Jan. 2, 1839, when the moon's distance from Aldebaran was $70° 45' 13''$.

1839. Jan. 2, 9^h Greenwich time,	Dist. $= 69° 26' 29''$
	P. L. $= 0.2852$
12^h	P. L. $= 0.2863$

Ans. $11^h 31^m 47^s$.

22. The correct distance of the moon from β Corvi, 1839, April 3d, $11^h 20^m$, in longitude $70°$ W. by account, was $54° 8' 15''$; what was the longitude ?

Solution.	$54° 8' 15''$	Gr. T. $= 11^h 20^m + 4^h 40^m = 16^h$
☽'s Dec. =	$26\ 48\ 52$	by N. A. sec. 0.04940
✱'s Dec. =	$22\ 30\ 11$	sec. 0.03439
$\frac{1}{2}$ sum =	$51° 43' 39''$	cos. 9.79198
Dist. — $\frac{1}{2}$ sum =	$2\ 24\ 36$	cos. 9.99961
		2)19.87538
	$3^h 59^m 43^s$	cos. 9.93769
✱'s R. A. =	$12\ 25\ 56$	
☽'s R. A. =	$16^h 25^m 37^s$	at Greenw. time $= 16^h$

Long. $= 16^h - 14^h 20^m = 4^h 40^m = 70°$, as supposed.

23. The correct distance of the moon from Castor, 1839, Nov.

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$29^d\ 19^h$, in longitude 45° W. by account, was 78° 3′; what was the longitude?

Greenwich, 1839,

Nov. $29^d\ 21^h$,	☽'s R. A. = $12^h\ 15^m\ 16^s.5$,	Dec. = 3° 48′ 31″ S.
22^h,	☽'s R. A. = 12 17 2 .9,	Dec. = 4 2 39 S.
Castor's	R. A. = 7 24 24 .4,	Dec. = 32 14 2 N.

Ans. 44° 18′ W.

24. Find the distance of the moon from the sun, 1839, August 12^d, Greenwich time at mean noon.

☉'s R. A. = $9^h\ 25^m\ 51^s.72$, Dec. = 15° 7′ 51″.5 N.

☽'s R. A. = 11 42 23 .48, Dec. = 0 57 27 .9 N.

Ans. 36° 33′ 14″.

25. Find the distance of the moon from the sun, 1839, August 14^d, Greenwich time at mean noon.

☉'s R. A. = $9^h\ 33^m\ 24^s.57$, Dec. = 14° 31′ 28″.2 N.

☽'s R. A. = 13 8 27 .62, Dec. = 10 25 54 .5 S.

Ans. 58° 50′ 38″.

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CHAPTER IX.

ABERRATION.

104. The apparent position of the stars is affected by two sources of optical deception, so that they are not in the direction in which they appear to be.

The first of these sources is the motion of the earth, and the corresponding correction is called *aberration.*

Aberration, like the earth's motion, is either *annual* or *diurnal.*

105. *Problem. To find the aberration of a star.*

Solution. The apparent direction of a star is obviously that of the telescope, through which the star is seen. Let S (fig. 47) be the star, and O the place of the observer at the instant of the observation; SO is the true direction of the star, or the path of the particle of light which proceeded from the star to the observer, and it would be the direction of the telescope if he were stationary. But if he is moving in the direction OP, the direction of the telescope OT must be such, that the end T was at the point R, in the line OS, at the same instant in which the particle of light was at this point. The length RT is, therefore, the distance gone by the observer while the light is describing the line OR.

If, then, we put

$V =$ the velocity of light,

$v =$ the earth's velocity,

$I = TOP = RTO,$

$\delta I = - ROT =$ the aberration from the true place,

$$m = \frac{v}{V \sin. 1''}, \tag{647}$$

we have,

$$V : v = OR : TR = \sin. I : -\delta I \sin. 1''$$

$$\delta I = -m \sin. I. \tag{648}$$

106. *Problem. To find the annual aberration in latitude and longitude.*

Solution. The earth is moving in the plane of the ecliptic at nearly right angles to the direction of the sun. Hence if TP (fig. 48) is the ecliptic, T the point towards which the earth is moving, S the true star, S' the apparent star,

$\odot$ = the sun's longitude,

A = the star's longitude, δA = the aberration in long.

L = the star's latitude, δL = the aberration in lat.

we have

$$ST = I, \;\; SP = L,$$

$$\text{long. of } T = \odot - 90°, \;\; PT = \odot - 90° - A = A_1$$

$$PP' = \delta A = TP - TP', \;\; \delta L = S'P' - SP$$

$$\cos. T = \text{cotan. } I \text{ tang. } A_1 = \text{cotan. } (I + \delta I) \text{ tang. } (A_1 - \delta A),$$

whence
$$\frac{\tan. (A_1 - \delta A)}{\text{tang. } A_1} = \frac{\text{tang. } (I + \delta I)}{\text{tang. } I}, \tag{649}$$

and, by (346 and 347),

$$\frac{\sin. \delta A}{\sin. (2 A_1 - \delta A)} = -\frac{\sin. \delta I}{\sin. (2 I + \delta I)}, \tag{650}$$

or omitting δA and δI in the denominators, and reducing by means of (648),

$$\delta A = -\frac{\sin. 2 A_1}{\sin. 2 I} \delta I = -\frac{\sin. A_1 \cos. A_1}{\sin. I \cos. I} \delta I$$

$$= m \frac{\sin. A_1 \cos. A_1}{\cos. I}. \tag{651}$$

But
$$\cos. I = \cos. A_1 \cos. L, \tag{652}$$

whence
$$\delta A = m \sin. A_1 \sec. L \tag{653}$$

$$= -m \cos. (\odot - A) \sec. L.$$

We also have

$$\sin. T = \frac{\sin. L}{\sin. I} = \frac{\sin. (L + \delta L)}{\sin. (I + \delta I)}, \tag{654}$$

whence

$$\sin. L \sin. (I + \delta I) = \sin. I \sin. (L + \delta L), \tag{655}$$

and

$$\begin{aligned} \delta L &= \frac{\sin. L \cos. I}{\cos. L \sin. I} \delta I \\ &= - m \tan g. L \cos. I \\ &= - m \cos. A_1 \sin. L \\ &= - m \sin. (\odot - A) \sin. L. \end{aligned} \tag{656}$$

107. *Problem. To find the annual aberration in distance and direction from the vernal equinox.*

Solution. Let A (fig. 48) be the vernal equinox, and let

$$M = SA, \quad \delta M = \text{aberration of } M$$

$$N = SAT, \; \delta N = \text{aberration of } N.$$

Now we have

$$\begin{aligned} \delta M = \delta I \cos. AST &= \frac{\sin. \odot - \cos. M \cos. I}{\sin. M \sin. I} \delta I \\ &= - m \frac{\sin. \odot - \cos. M \cos. I}{\sin. M}. \end{aligned} \tag{657}$$

But

$$\cos. I = \sin. \odot \cos. M - \cos. \odot \sin. M \cos. N, \tag{658}$$

whence if we put

$$B = - m \sin. \odot \tag{659}$$

$$C = - m \cos. \odot, \tag{660}$$

we have

$$\delta M = B \sin. M + C \cos. M \cos. N.$$

Again; the triangles ASS' and ATS' give by (302),

$$\sin. AST = \frac{\sin. M \,.\, \delta N}{\delta I} = - \frac{\cos. \odot \sin. N}{\sin. I} \tag{661}$$

$$\delta N = m \cos. \odot \frac{\sin. N}{\sin. M} = - \frac{C \sin. N}{\sin. M}. \tag{662}$$

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108. *Problem. To find the annual aberration in right ascension and declination.*

Solution. If AT (fig. 48) were the equator, we should have

$$D = SP, \quad R = AP,$$

and if we put

$$N_1 = SAP, \quad \omega = \text{obliquity of ecliptic},$$

we have

$$N_1 = N + \omega,$$

and the triangles ASP, $AS'P'$ give

$$\sin. D = \sin. M \sin. N_1 \tag{663}$$

$$\sin. (D - \delta D) = \sin. (M - \delta M) \sin. (N_1 - \delta N) \tag{664}$$

$$\begin{aligned} \cos. D \, \delta D &= \sin. M \cos. N_1 \, \delta N + \cos. M \sin. N_1 \, \delta M \\ &= B \sin. M \cos. M \sin. N_1 \\ &\quad - C (\sin. N \cos. N_1 - \cos.^2 M \sin. N_1 \cos. N), \end{aligned} \tag{665}$$

and if we put

$$A - C \cos. \omega \tag{666}$$

$$b' = \sin. M \cos. M \sin. N_1 \sec. D \tag{667}$$

$$a' = -(\sin. N \cos. N_1 - \cos.^2 M \sin. N_1 \cos. N) \sec. D \sec. \omega, \tag{668}$$

we have

$$\cos. M = \cos. D \cos. R \tag{669}$$

$$\text{cotan}. N_1 = \sin. R \,\text{cotan}. D \tag{670}$$

$$\begin{aligned} \sin. M \cos. N_1 &= \frac{\sin. D \cos. N_1}{\sin. N_1} = \sin. D \,\text{cotan}. N_1 \\ &= \sin. D \sin. R \,\text{cotan}. D = \sin. R \cos. D \end{aligned} \tag{671}$$

$$b' = \sin. D \cos. R \cos. D \sec. D = \sin. D \cos. R \tag{672}$$

$$\begin{aligned} a' &= -[\sin. (N - N_1) + \sin.^2 M \sin. N_1 \cos. N] \sec. D \sec. \omega \\ &= [\sin. \omega - \sin.^2 M \sin.^2 N_1 \sin. \omega] \sec. D \sec. \omega \\ &\quad - \sin.^2 M \sin. N_1 \cos. N_1 \cos. \omega \sec. D \sec. \omega \\ &= (1 - \sin.^2 D) \sin. \omega \sec. D \sec. \omega - \sin. M \sin. D \cos. N_1 \sec. D \\ &= \cos. D \tan. \omega - \sin. R \sin. D \end{aligned} \tag{673}$$

$$\delta D = A a' + B b'. \tag{674}$$

Again, we have

$$\cos. M = \cos. R \cos. D \tag{675}$$

$$\cos. (M + \delta M) = \cos. (R + \delta R) \cos. (D + \delta D)$$

$$\cos. D \sin. R \,\delta R = \sin. M \,\delta M - \cos. R \sin. D \,\delta D$$

$$= B (\sin.^2 M - b' \cos. R \sin. D)$$

$$+ A (\sin. M \cos. M \cos. N \sec. \omega - a' \cos. R \sin. D), \tag{676}$$

and if we put

$$a = (\sin. M \cos. M \cos. N \sec. \omega - a' \cos. R \sin. D) \sec. D \operatorname{cosec.} R$$

$$b = (\sin.^2 M - b' \cos. R \sin. D) \sec. D \operatorname{cosec.} R,$$

we have

$$a \cos. D \sin. R = \sin. M \cos. M \cos. N_1 + \sin. R \cos. R \sin.^2 D$$

$$+ (\sin. M \cos. M \sin. N_1 - \cos. R \sin. D \cos. D) \tan. \omega$$

$$= \sin. R \cos. R (\cos.^2 D + \sin.^2 D)$$

$$+ (\sin. M \sin. N_1 \cos. R \cos. D - \cos. R \sin. M \sin. N_1 \cos. D) \tan. \omega$$

$$= \sin. R \cos. R$$

$$a = \cos. R \sec. D \tag{677}$$

$$b \cos. D \sin. R = 1 - \cos.^2 M - \sin.^2 D \cos.^2 R$$

$$= 1 - \cos.^2 D \cos.^2 R - \sin.^2 D \cos.^2 R$$

$$= 1 - \cos.^2 R = \sin.^2 R$$

$$b = \sin. R \sec. D \tag{678}$$

$$\delta R = A a + B b, \tag{679}$$

and formulas (659, 660, 672, 673, 674, 677, 678, 679) agree with those given in the Nautical Almanac for finding the annual aberration.

109. *Corollary.* The value of m, which is used in the Nautical Almanac, is

$$m = 20''.3600,$$

which gives

$$m \cos. \omega = 20''.3600 \cos. 23° \, 27' \, 36''.98 = 18''.6768.$$

110. *Scholium.* In the values of the aberration in right ascension and declination, each term consists of two factors, one of which is the

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same each instant for all the stars, and the other is the same for each star, during several years.

111. *Corollary.* If in (674) and (679) we put

$$i = A \tan.\, \omega \tag{680}$$

$$B = h \cos.\, H \tag{681}$$

$$A = h \sin.\, H; \tag{682}$$

they become

$$\delta\, D = i \cos.\, D - h \sin.\, H \sin.\, R \sin.\, D + h \cos.\, H \cos.\, R \sin.\, D$$

$$= i \cos.\, D + h \cos.\, (H + R) \sin.\, D \tag{683}$$

$$\delta\, R = h \sin.\, H \cos.\, R \sec.\, D + h \cos.\, H \sin.\, R \sec.\, D$$

$$= h \sin.\, (H + R) \sec.\, D, \tag{684}$$

which agree with the formulas in the Nautical Almanac.

112. We have from (659–679)

$$\delta\, R = \sec.\, D\, [- m \cos.\, \omega \cos.\, \odot \cos.\, R - m \sin.\, \odot \sin.\, R] \tag{685}$$

$$= \sec.\, D\, [- \tfrac{1}{2}\, m\, (\cos.\, \omega + 1)\, (\cos.\, \odot \cos.\, R + \sin.\, \odot \sin.\, R)$$

$$+ \tfrac{1}{2}\, m\, (1 - \cos.\, \omega)\, (\cos.\, \odot \cos.\, R - \sin.\, \odot \sin.\, R)]$$

$$= \sec.\, D\, [- m \cos.^2 \tfrac{1}{2}\, \omega \cos.\, (R - \odot) + m \sin.^2 \tfrac{1}{2}\, \omega \cos.\, (R + \odot)],$$

and if we put

$$Q = R - \odot, \quad Q' = R + \odot \tag{686}$$

$$n = - m \cos.^2 \tfrac{1}{2}\, \omega, \quad n' = m \sin.^2 \tfrac{1}{2}\, \omega, \tag{687}$$

(645) becomes

$$\delta\, R = \sec.\, D\, (n \cos.\, Q + n' \cos.\, Q'), \tag{688}$$

and the values of $n \cos.\, Q$ and $n' \cos.\, Q'$ may be put in tables like Parts I and II of Table XLII of the Navigator.

Again, we have

$$\delta\, D = \sin.\, D\, (m \cos.\, \omega \sin.\, R \cos.\, \odot - m \cos.\, R \sin.\, \odot)$$

$$- m \sin.\, \omega \cos.\, \odot \cos.\, D$$

$$= \sin. D \left[\tfrac{1}{2} m (\cos. \omega + 1) \sin. Q - \tfrac{1}{2} m (1 - \cos. \omega) \sin. Q'\right]$$
$$- \tfrac{1}{2} m \sin. \omega \left[\cos. (\odot + D) + (\cos. \odot - D)\right]$$
$$= \sin. D \left[- m \cos.^2 \tfrac{1}{2} \omega (\cos. Q + 90^\circ) + \tfrac{1}{2} m \sin.^2 \tfrac{1}{2} \omega \cos. (Q' + 90^\circ)\right]$$
$$- \tfrac{1}{2} m \sin. \omega \left[\cos. (\odot + D) + \cos. (\odot - D)\right]$$
$$= \sin. D \left[- n \cos. (Q + 90^\circ) + n' \cos. (Q' - 90^\circ)\right]$$
$$- \tfrac{1}{2} m \sin. \omega \left[\cos. (\odot + D) + \cos. (\odot - D)\right], \qquad (689)$$

and the values of

$$- \tfrac{1}{2} m \sin. \omega \cos. (☉ + D) \text{ and } - \tfrac{1}{2} m \sin \omega \cos. (☉ - D)$$

may be put in a table like Part III of Table XLII. The rules for finding the variations in right ascension and declination are then the same as in the explanation of this table.

113. In constructing Table XLII, the values of m and ω were taken

$$m = 20'', \quad \omega = 23^\circ\, 27'\, 28'', \qquad (690)$$

whence

$$n = - 19''.173, \quad n' = 0''.827, \qquad (691)$$
$$- \tfrac{1}{2} m \sin. \omega = - 3''.9814. \qquad (692)$$

114. By putting

$$☉ - \varLambda = P, \qquad (693)$$

we have, by (653 and 656),

$$\delta L = - m \cos. (P - 90^\circ) \sin. L \qquad (694)$$
$$\delta \varLambda = - m \cos. P \sec. L, \qquad (695)$$

so that if the values of

$$- m \cos. P$$

are inserted in tables like Table XLI of the Navigator, the variations of latitude and longitude are found by the rule given in the explanation of this table.

115. If the star is nearly in the ecliptic, the aberration in latitude may be neglected, and the aberration in longitude will be by (695)

$$\delta \varLambda = - m \cos. P \qquad (696)$$

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116. *Problem. To find the diurnal aberration in right ascension and declination.*

Solution. Let

$v' =$ the velocity of a point of the equator, arising from the earth's rotation,

$$m' = \frac{v'}{V \sin. 1''}. \tag{697}$$

The velocity of the observer is evidently in proportion to the circumference which he describes in a day, that is, to the radius of this circumference, or to the cosine of the latitude.

The velocity of the observer $= v' \cos.$ lat.

Now, the diurnal motion is parallel to the equator, whence the formulas (653) and (656) may be referred at once to the present case by putting

$Z =$ the right ascension of the zenith,

and changing m into $m' \cos.$ lat., $\odot - A$ into $Z - R$, and L into D; whence the diurnal aberrations in right ascension and declination are

$$\delta' R = - m' \cos. (Z - R) \sec. D \cos. \text{lat.} \tag{698}$$

$$\delta' D = - m' \sin. (Z - R) \sin. D \cos. \text{lat.} \tag{699}$$

117. The value of m' is nearly

$$m' = 0''.31. \tag{700}$$

118. *Problem. To find the aberration which arises from the motion of a planet.*

Solution. The most important planets revolve about the sun almost uniformly in circles, and in the plane of the ecliptic. At the instant, then, of the light's reaching the earth, the planet has advanced in its orbit by a distance proportioned to its velocity, and to the time which the light takes in reaching the earth. Let then S (fig. 49) be the sun, and $O_1 O_1'$ perpendicular to $O_1 S$ the path of the planet; and put

$v_1 =$ the velocity of the plane,

$$m_1 = \frac{v_1}{V \sin. 1''}, \quad P_1 = OO_1S,$$

$$r = OS, \quad r_1 = O_1S,$$

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we have

$$\delta_1 A = - O_1 O O_1' = - \frac{O_1 O_1' \text{ cos. } P_1}{O O_1 \text{ sin. } 1''} = - m_1 \text{ cos. } P_1. \quad (701)$$

But it will be shown in Theoretical Astronomy that

$$v^2 : v_1^2 = r_1 : r;$$

hence

$$m^2 : m_1^2 = v^2 : v_1^2 = r_1 : r$$

$$m : m_1 = \sqrt{r_1} : \sqrt{r}$$

$$m_1 = m \sqrt{\frac{r}{r_1}} \quad (703)$$

$$\delta_1 A = - m \sqrt{\frac{r}{r_1}} \text{ cos. } P_1; \quad (703)$$

and this aberration being combined with (696) gives the whole aberration in longitude, from which a table, like Table XXXIX of the Navigator, may be constructed.

119. Examples.

1. Find the values of log. A, log. B, h, H, and i for May 1, 1839, when $\odot = 40° \, 52' \, 56''$.

Ans. log. $A = 1.1498_n$
log. $B = 1.1248_n$
$h = 19''.42$
$H = 226° \, 40'$
$i = - 6''.13$

2. Find the values of log. a, log. b, log. a', log. b' for Altair in the year 1839.

Solution.

$R = 19^h \, 42^m \, 55^s$	cos. 9.63760		sin. 9.95466$_n$
$D = 8° \, 26' \, 52''$	sec. 0.00474		sec. 0.00474
	log. a = 9.64234		log. b = 9.95940$_n$
R cos. 9.63760		sin. 9.95466$_n$	
D sin. 9.16704		sin. 9.16704	cos. 9.99526
log. b' = 8.80464	0.13234	9.12170^n	ω tan. 9.63747
	0.42927		9.63273
	a' = 0.56161		log. a' = 9.74944

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3. Find the values of log. a, log. b, log. a', log. b', for Regulus in the year 1839, for this star,

$$R = 9^h\ 59^m\ 48^s, \quad D = 12^\circ\ 45'\ 7''.$$

Ans. log. $a = 9.94816\,n$

log. $b = 9.71048$

log. $a' = 9.49516$

log. $b' = 9.28122\,n$

4. Find the numbers of the different parts of Table XLII for the argument $7^s\ 20^\circ = 230^\circ$.

Ans. $12''.32$ for Part I,

$-0''.53$ for Part II,

$2''.56$ for Part III.

5. Find the number of Table XLI for $7^s\ 20^\circ$.

Ans. $12''.9$.

6. Find the aberration in right ascension and declination of Altair for May 1, 1839.

Solution. I.

	A 1.1498 n		1.1498 n
	a 9.6423		a' 9.7494
$-6''.20$	0.7921 n	$-7''.93$	0.8992 n
	B 1.1248 n		1.1248 n
	b 9.9594 n		b' 8.8046
$12''.14$	1.0842	$-0''.85$	9.9294 n
$\delta R = 5''.9 = 0^s.39$		$\delta D = -8''.78$	

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II.

$H + \alpha = 162° \, 44' + 360°$	sin. 9.4725		cos. 9.9800 n
$h = 19''.42$	1.2882		1.2882
$D = 8° \, 27'$	sec 0.0047		sin. 9.1670
$\delta R = 5''.83 = 0^s.39$	0.7654	$-2''.72$	0.4352 n
	i cos. $D =$	$-6''.06$	
	$\delta D =$	$-8''.78$	

III.

$R - \odot = 255° \, 40' = 8^s \, 15° \, 50'$	P. I. $= 4''.75$		
$R + \odot = 76° + 360 = 2^s \, 16° + 12^s$	P. II. $= 0''.20$		
	$4''.95$		0.6946
	D		sec. 0.0047
$\delta R = 5'' = 0^s.33$			0.6993
$8^s \, 15° \, 40' + 3^s = 11^s \, 15° \, 40'$	P. I. $-$ $18''.57$		
$2^s \, 16° + 3^s = 5^s \, 16°$	P. II. $-$ $0''.80$		
	$-19''.37$		1.2871 n
	D		sin. 9.1670
	$-2''.85$		0.4541 n
$\odot + D = 48° = 1^s \, 18°$	$-2''.66$		
$\odot - D = 32° = 1^s \; 2°$	$-3''.38$		
$\delta D =$	$-8''.89$		

7. Find the aberration in right ascension and declination of Regulus for May 1, 1839.

Ans. By Naut. Alm. $\delta R = 0^s.38$
$\delta D = -1''.87$
By the Navigator $\delta R = 0^s.38$
$\delta D = -1''.91$

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8. Find the aberration of Regulus in latitude and longitude for May 1, 1839.

Ans. $\delta A = 6''.5$
$\delta D = 0''.15.$

9. Find the aberration of Venus in longitude, when the difference of longitude of Venus and the sun is 45°.

Solution.	r	0.0000		0.0000
	r_1	ar. co. 0.1407	$\frac{1}{2}$ (ar. co.)	0.0703
$P = 45°$		sin. 9.8495	$20''$ log.	1.3010
$P_1 =$		sin. 9.9902	cos.	9.3214
				0.6927

— $5''$ when P_1 is acute, + $5''$ when P_1 is obtuse,
— $14''$ from Table XLI, — $14''$

$\delta A = -19''$ when P_1 is ac., $= -\ 9''$ when P_1 is obtuse.

10. Find the aberration of each of the planets in longitude, when the difference of longitude of the sun and planet is 15°. The value of log. r_1 for each of the planets is

For Mercury	9.5878	is the mean value,
Venus	9.8593	
The Earth	0.0000	
Mars	0.1829	
Jupiter	0.7161	
Saturn	0.9795	
Uranus	1.2829	

Ans.

For Mercury	$-43''$ when P_1 is acute,
	$4''$ when P_1 is obtuse,
Venus	$-41''$ when P_1 is acute,
	$3''$ when P_1 is obtuse,
Mars	$35''$
Jupiter	$28''$
Saturn	$26''$
Uranus	$24''$

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11. Find the diurnal aberration of right ascension and declination of Polaris for Jan. 1, 1839, and latitude 45°, when the hour angle is $0^h\ 30^m$.

Solution.	$0''.31$	9.4914	9.4914
	$45°$	cos. 9.8495	9.8495
$D = 88°\ 27'$		sec. 1.5678	sin. 9.9998
$0^h\ 30^m$		cos. 9.9963	sin. 9.1157
$\delta' R = -8''.04 = -0^s.53$		0.9050	$\delta' D = 0''.03$ 8.4564

12. Find the diurnal aberration of δ Ursæ Minoris in right ascension and declination for Jan. 1, 1839, and latitude 0°, when the star is upon the meridian.

Dec. of δ Ursæ Minoris $= 86°\ 35'$.

Ans. $\delta' R = -0^s.35$

$\delta' D = 0.$

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CHAPTER X.

REFRACTION.

120. Light proceeds in exactly straight lines, only in the void spaces of the heavens; but when it enters the atmosphere of a planet, it is sensibly bent from its original direction according to known optical laws, and its path becomes curved. This change of direction is called *refraction;* and the corresponding change in the position of each star is the *refraction of that star.*

121. *Problem. To find the refraction of a star.*

Solution. Let O (fig. 50) be the earth's centre, A the position of the observer, AOK the section of the surface formed by a vertical plane passing through the star. It is then a law of optics, that

Astronomical Refraction takes place in vertical planes, so as to increase the altitude of each star without affecting its azimuth.

Let, now, ZIH be the section of the upper surface of the upper atmosphere formed by the vertical plane, SI the direction of the ray of light which comes to the eye of the observer. This ray begins to be bent at I, and describes the curve IA, which is such, that the direction AC is that at which it enters the eye. Let, now,

$$\varphi = ZAC = \text{the } \ast\text{'s apparent zenith distance,}$$

$$r = \text{the refraction,}$$

$$= \text{the diff. of directions of } AC \text{ and } IS,$$

$$= SIL - S'CL$$

$$u = COZ,$$

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and we have

$$LCS' = \varphi - u,$$

$$SIL = \varphi - u + r.$$

Again, it is a law of optics that the *ratio of the sines of the two angles LIS and ZAS' is constant for all heights, and dependent upon the refractive power of the air at the observer.*

Denote this ratio by n, and we have

$$\frac{\sin.\ (\varphi - u + r)}{\sin.\ \varphi} = n, \tag{704}$$

and if

U and R = the values of u and r at the horizon,

we have

$$\frac{\sin.\ (\varphi - u + r)}{\sin.\ \varphi} = n = \cos.\ (U - R), \tag{705}$$

whence

$$\frac{\sin.\ \varphi - \sin.\ (\varphi - u + r)}{\sin.\ \varphi + \sin.\ (\varphi - u + r)} = \frac{1 - \cos.\ (U - R)}{1 + \cos.\ (U - R)} \tag{706}$$

$$\frac{\text{tang.}\ \frac{1}{2}\ (u - r)}{\text{tang.}\ [\varphi - \frac{1}{2}\ (u - r)]} = \text{tang.}^2\ \tfrac{1}{2}\ (U - R) = N, \tag{707}$$

and since $\frac{1}{2}\ (u - r)$ is small,

$$\tfrac{1}{2}\ (u - r) = N\ \text{tang.}\ [\varphi - \tfrac{1}{2}\ (u - r)]. \tag{708}$$

Again, to find u, the triangle COA gives

$$\frac{\sin.\ (\varphi - u)}{\sin.\ \varphi} = \frac{OA}{OC}. \tag{709}$$

Now the point C is at different heights for different zenith distances of the star; but this difference in the values of OC is small, and may be neglected in this approximation; so that

$$\frac{\sin.\ (\varphi - u)}{\sin.\ \varphi} = \cos.\ U = \frac{OA}{OK}, \tag{710}$$

$$\frac{\sin.\ \varphi - \sin.\ (\varphi - u)}{\sin.\ \varphi + \sin.\ (\varphi - u)} = \frac{1 - \cos.\ U}{1 + \cos.\ U}, \tag{711}$$

$$\tan.\ \tfrac{1}{2}\ u = \text{tang.}^2\ \tfrac{1}{2}\ U\ \tan.\ (\varphi - \tfrac{1}{2}\ u). \tag{712}$$

and since u is small,

$$\tfrac{1}{2}\, u = \text{tang.}^2\, \tfrac{1}{2}\, U \,\text{tang.}\, (\varphi - \tfrac{1}{2}\, u), \tag{713}$$

which, compared with this rough value of $\frac{1}{2}(u - r)$ from (708),

$$\tfrac{1}{2}\,(u - r) = N \,\text{tan.}\, (\varphi - \tfrac{1}{2}\, u) \tag{714}$$

gives

$$u = \frac{r}{1 - N \cot.^2 \frac{1}{2}\, U} = N'\, r, \tag{715}$$

and if we put

$$m = \frac{2\,N}{N' - 1} \tag{716}$$

$$p = \tfrac{1}{2}\,(N' - 1), \tag{717}$$

we have, by (708),

$$\tfrac{1}{2}\,(u - r) = p\, r \tag{718}$$

$$r = m \,\text{tan.}\, (\varphi - p\, r), \tag{719}$$

and the values of m and p must be determined by observation; and their mean values, as found by Bradley, and adopted in the Navigator, are

$$m = 57''.035, \quad p = 3, \tag{720}$$

by which Table XII is calculated.

122. The variation in the values of m and p for different altitudes of the star, can only be determined from a knowledge of the curve which the ray of light describes. But this curve depends upon the law of the refractive power of the air at different heights; and this law is not known, so that the variations of m and p must be determined by observation. At altitudes greater than 12 degrees, the mean values of m and p are found to be nearly constant, and observations at lower altitudes are rarely to be used.

123. The mean values of m and p, which are given in (720), correspond to

the height of the barometer $= 29.6$ inches, (721)

the thermometer $= 50°$ Fahrenheit. (722)

Now the refraction is proportional to the density of the air; but, at the same temperature, the density of the air is proportional to its elastic power, that is, to the height of the barometer. If then

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$h =$ the height of the barometer in inches,

$r =$ the refraction of Table XLI,

$\delta r =$ the correction for the barometer;

we have

$$r : r + \delta r = 29.6 : h \tag{723}$$

$$29.6\, \delta r = (h - 29.6)\, r \tag{724}$$

$$\delta r = \frac{(h - 29.6)}{29.6}\, r, \tag{725}$$

whence the corresponding correction of Table XXXVI is calculated.

Again, the density of the air, for the same elastic force, increases by one four-hundredth part for every depression of 1° of Fahrenheit; hence the refraction increases at the same rate, so that if

$\delta' r =$ the correction for the thermometer,

$f =$ the temperature in degrees of Fahrenheit,

we have

$$\delta' r = \frac{50 - f}{400}\, r, \tag{726}$$

whence the corresponding correction of Table XXXVI is calculated.

124. Examples.

1. Find the refraction, when the altitude of the star is 14°, and the corrections for this altitude, when the barometer is 31.32 inches, and the thermometer 72° Fahrenheit.

Solution. $57''.035$ log.	1.75614			
76° tan.	0.60323			
1st app. $r = 228''.7 = 3'\,48''.7$	2.35937			
$57''.035$	1.75614			
$76° - 3r = 75°\,48'\,34''$ tan.	0.59711			
2d app. $r = 226'' = 3'\,46''$	2.35325			2.353
$31.32 - 29.6 = 1.72$	0.235		$50 - 72 = -22$	1.342n
29.6 ar. co.	8.529		400 ar. co.	7.398
$\delta r = 13''$	1.117		$\delta' r = -12''$	1.093

2. Find the refraction, when the altitude of the star is 50°, and the corrections for this altitude, when the barometer is 31.66 inches, and the thermometer 36°.

Ans. The refraction $= 48''$
Correction for barometer $= 3''$
Correction for thermom. $= 2''$.

3. Find the refraction, when the altitude of the star is 10°, and the corrections for this altitude, when the barometer is 27.80 inches, and the thermometer 32°.

Ans. The refraction $= 5'\ 15''$
Correction for barometer $= -19''$
Correction for thermom. $= 15''$.

125. *Problem. To find the radius of curvature of the path of the ray of light in the earth's atmosphere.*

Solution. By the *radius of curvature* is meant the radius of the circular arc, which most nearly coincides with the curve. Now this radius may be found with sufficient accuracy, by regarding the whole curve AB as the arc of a circle; and if we put

$$r_1 = \text{the radius of curvature,}$$

$$R_1 = OA = \text{the earth's radius,}$$

we have

$$AC : R_1 = \sin. u : \sin. (\varphi + u), \tag{727}$$

or, nearly,

$$AB : R_1 = u \sin. 1'' : \sin. \varphi$$

$$AB = \frac{R_1\, u \sin. 1''}{\sin. \varphi}. \tag{728}$$

Again, the radii of the arc AB, which are drawn to the points A and B, are perpendicular to the tangents AS' and BS, so that the angle which they make with each other is

$$S'AS = r;$$

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that is, r is the angle at the centre, which is measured by the arc AB, consequently

$$AB = r_1 \sin. r = r_1\, r \sin. 1'', \tag{729}$$

whence

$$r_1 = \frac{u\, R_1}{r \sin. \varphi}. \tag{730}$$

But, by (718),

$$u = 7\, r, \tag{731}$$

whence

$$r_1 = \frac{7\, R_1}{\sin. \varphi}, \tag{732}$$

so that at the horizon

$$r_1 = 7\, R_1, \tag{733}$$

as in (284, 285).

126. *Problem. To find the dip of the horizon.*

Solution. The dip of the horizon is the error of supposing the apparent horizon to be only 90° from the zenith, whereas it is more than 90°. If O (fig. 51) is the centre of the earth, B the position of the observer at the height AB above the surface, O' the centre of curvature of the visual ray BT, which just touches the earth's surface at T, BT' perpendicular to $O'B$, is the direction of the apparent horizon, and

$$\delta\, H = HBT' = OBO' = \text{the dip.}$$

The triangle BOO' gives

$$BO' : OO' = \sin. BOO' : \sin. \delta\, H = \sin. BOT' : \sin. \delta\, H,$$

or, since $BO' = 7\, BO$ nearly, and $OO' = 6\, BO$,

and $\delta\, H$ and BOT' are small,

$$7 : 6 = BOT' : \delta\, H$$

$$\delta\, H = \tfrac{6}{7}\, BOT' = \tfrac{6}{7} \frac{AT'}{AO \sin. 1''}. \tag{734}$$

But, by (285), we have, if we put

$$R = AO,\ h = AB$$

$$\tfrac{6}{7}\, AT' = \tfrac{6}{7} \surd (\tfrac{7}{3}\, R\, h)$$

$$= 2 \surd (\tfrac{3}{7}\, R\, h) \tag{735}$$

whence

$$\delta H = \frac{2}{\sqrt{(\frac{7}{3} R)} \sin. 1''} \sqrt{h}, \tag{736}$$

and

$$\log. \delta H = \log. 2 - \log. (\sqrt{\tfrac{7}{3}} R) - \log. \sin. 1'' + \tfrac{1}{2} \log. h$$
$$= 1.77128 + \tfrac{1}{2} \log. h, \tag{737}$$

which is the same with the formula, given in the preface to the Navigator, for calculating Table XIII.

127. *Problem. To find the dip of the sea at different distances from the observer.*

Solution. Let O (fig. 52) be the centre of the earth, B the observer at the height

$$h = AB \text{ (in feet)}$$

above the sea, and A' the point of the sea which is observed at the distance

$$d = AA' \text{ (in sea miles)} = AOA'$$

from B; and let

$$M = \text{the length of a sea mile in feet.}$$

If the radius OA' is produced to B', so that

$$A'B' = AB,$$

the point B' will be elevated by refraction nearly as much as the point A'. But the visual ray BB' will, from the equal heights of B and B', be perpendicular to the radius OC, which is half way between B and B', so that the dip of B' is, by (734),

$$\delta B = \tfrac{6}{7} BOC = \tfrac{3}{7} AOA' = \tfrac{3}{7} d. \tag{738}$$

The dip of the point A' will be greater than B' by the angle

$$i = B'BA,$$

which it subtends at B, and which is found with sufficient accuracy by the formula

$$\sin. i = \frac{A'B'}{A'B} = \frac{h}{M\, d} = i \sin. 1' \tag{739}$$

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$$i = \frac{h}{M \sin. 1' d}. \tag{740}$$

But, by (286),

$$M = \frac{\pi R}{10800'}. \tag{741}$$

$$\frac{1}{M \sin. 1'} = \frac{10800'}{\pi R \sin. 1'} = 0.56514, \tag{742}$$

sa that the dip of A' is

$$\delta A = \tfrac{3}{7} d + 0.56514 \frac{h}{d}, \tag{743}$$

which is the same with the formula, given in the preface to the Navigator, for calculating Table XVI.

128. Refraction, by elevating the stars in the horizon, will affect the times of their rising and setting; and the star will not set until its zenith distance is

$$90^\circ + \text{horizontal refraction},$$

and the corresponding hour angle is easily found by solving the triangle *PZB* (fig. 35).

129. Another astronomical phenomenon, connected with the atmosphere, and dependent upon the combination of reflection and refraction, is the *twilight*, or the light before and after sunset, which arises from the illuminated atmosphere in the horizon. This light begins and ends when the sun is about 18° below the horizon; so that the time of its beginning or ending is easily calculated from the triangle *PZB* (fig. 35).

130. Examples.

1. Find the dip of the horizon, when the height of the eye is twenty feet.

Ans. $264'' = 4' 24''$.

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2. Find the dip of the sea at the distance of 3 miles, when the height of the eye is thirty feet.

Solution.

$$\tfrac{3}{7} \times 3 = \tfrac{9}{7} = 1'.3$$
$$0.56514 \times \tfrac{30}{3} = 5'.6$$
$$\text{dip} = 7'.$$

3. Find the dip of the sea at the distance of $2\frac{1}{2}$ miles, when the height of the eye is forty feet.

Ans. 10′.

4. Find the dip of the sea at the distance of $\frac{1}{4}$ of a mile, when the height of the eye is thirty feet.

Ans. 68′.

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CHAPTER XI.

PARALLAX.

131. THE fixed stars are at such immense distances from the earth, that their apparent positions are the same for all observers. But this is not the case with the sun, moon, and planets; so that, in order to compare together observations taken in different places, they must be reduced to some one point of observation. The point of observation which has been adopted for this purpose, is the earth's centre; and the difference between the apparent positions of a heavenly body, as seen from the surface or the centre of the earth, is called its *parallax.*

132. *Problem. To find the parallax of a star.*

Solution. Let O (fig. 53) be the earth's centre, A the observer, S the star, and OSA, being the difference of directions of the visual rays drawn to the observer and the earth's centre, is the parallax. Now since SAZ is the apparent zenith distance of the star, and SOZ is its distance from the same zenith to an observer at O, the parallax

$$OSA = p$$

is the excess of the apparent zenith distance above the true zenith distance. If, then,

$z = SAZ$, $R = OA =$ the earth's radius,

$r = OS =$ the distance of the star from the earth's centre,

we have
$$r : R = \sin. z : \sin. p,$$

or
$$\sin. p = \frac{R \sin. z}{r}, \tag{744}$$

or
$$p = \frac{R \sin. z}{r \sin. 1''}. \tag{745}$$

133. *Corollary.* If P is the horizontal parallax, we have

$$\sin. P = \frac{R}{r}, \tag{746}$$

or

$$P = \frac{R}{r \sin. 1''}; \tag{747}$$

whence

$$\sin. p = \sin. P . \sin. z, \tag{748}$$

or

$$p = P . \sin. z, \tag{749}$$

which agrees with (604) and Tables X A., XIV, and XXIX, are computed by this formula, combined, in the last Table, with the refraction of Table XII.

134. *Corollary.* In common cases, the value of the horizontal parallax can be taken from the Nautical Almanac; but, in eclipses and occultations, regard must be had to the length of the earth's radius, which is different for different places. The earth is not a sphere, but *a spheroid slightly compressed at the poles; the polar radius being less than the equatorial one by about* $\frac{1}{300}$th *part.* The spheroid may be obtained from the sphere by such a compression over the whole surface parallel to the polar axis, *that each place is brought nearer to the plane of the equator by* $\frac{1}{300}$th *part.*

Thus, if $OEAP$ (fig. 54) is a section of the earth through the polar axis OP and $OEA'P'$, the section of the sphere of which the equatorial semidiameter OE is the radius; and if $A'AM$, $B'BN$, are drawn parallel to OP, each of the distances $A'A$, $B'B$, $P'P$, &c., will be $\frac{1}{300}$th part of the distances $A'M$, $B'N$, $P'O$, &c.

135. *Problem. To find the reduction of parallax.*

The horizontal parallax is, by (747), proportional to the earth's radius, so that it diminishes at the same rate, from the equatorial value which is given in the Nautical Almanac. Hence, if AR is drawn perpendicular to OA,

$$L'' = A'OL,$$

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$\delta R =$ the diminution of R for the latitude L,

$\delta P =$ that of P,

$R =$ the radius at the equator,

$P =$ the parallax at the equator,

$m = \frac{1}{300}$,

we have

$$A'M = OA' \sin. A'OM = R \sin. L''$$

$$AA' = m\ R \sin. L'' = m\ R \sin. L \text{ nearly}$$

$$\delta R = A'R \text{ nearly}$$

$$= AA' \sin. A'AR = m\,R \sin.^2 L$$

$$= \tfrac{1}{300}\ R \sin.^2 L$$

$$= \tfrac{1}{600}\ R\,(1 - \cos. 2\,L) \tag{750}$$

$$\delta P = \tfrac{1}{300}\ P \sin.^2 L$$

$$= \tfrac{1}{600}\ P\,(1 - \cos. 2\,L), \tag{751}$$

and if P is expressed in minutes, while δP is expressed in seconds, (751) becomes

$$\delta P \text{ in seconds} = \tfrac{1}{10}\ (P \text{ in minutes})\ (1 - \cos. 2\,L), \tag{752}$$

which agrees with the formulas for calculating the reduction of parallax given in the explanation to Table XXXVIII of the Navigator.

136. In reducing delicate observations to the centre of the earth, it must be observed that the centre is not exactly in the direction of the vertical. Thus, if A is the observer, Z the zenith, ZAL the vertical, Z' the point where the radius OA produced meets the celestial sphere, Z' is called the *true zenith*, and Z the *apparent zenith*. The angle ZAZ', which is the difference between the polar distance of the true and apparent zenith, is called the *reduction of the latitude*, and must be subtracted from the angle ALE, or the latitude to obtain the angle AOE, or the direction of the observer from the earth's centre. The angle AOE is called *the reduced latitude*, and is to be substituted for the latitude in reducing delicate observations to the centre of the earth.

137. *Problem. To find the reduction of the latitude.*

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Solution. Draw AC and $A'C'$ (fig. 54) parallel to OE; and since AB is perpendicular to AL, the angle

$$L = ALE = CBA.$$

Let
$$L' = AOE,$$
and
$$\delta L = L - L'$$
is the reduction of the latitude.

Let also
$$x = OM, \ x' = ON$$
$$y = A'M, \ y' = B'N$$
$$n = 1 - m = \tfrac{299}{300} \tag{753}$$
so that
$$AM = n\,y, \ BN = n\,y'$$
we have their terms

$$\text{tang. } L'' = \frac{A'C'}{B'C'} = \frac{x - x'}{y - y'} = \frac{A'M}{MO} = \frac{y}{x} \tag{754}$$

$$\text{tang. } L = \frac{AC}{BC} = \frac{x - x'}{n\,(y - y')} = \frac{1}{n} \text{ tang. } L'' \tag{755}$$

$$\text{tang. } L' = \frac{AM}{MO} = \frac{n\,y}{x} = n \text{ tang. } L'' \tag{756}$$

$$\frac{\text{tan. } L}{\text{tan. } L'} = \frac{1}{n^2} = \left(\frac{300}{299}\right)^2 = 1.0067001$$

$$= \frac{\text{tang. } L}{\text{tang. } (L - \delta L)}, \tag{757}$$

which agrees with the formula given in the explanation of Table XXXVIII in the Navigator, and which must be computed by means of tables of 7 places of decimals.

138. *Corollary.* By applying (346 and 347) to (757), we obtain

$$\frac{\sin.\,\delta L}{\sin.\,(2\,L - \delta L)} = \frac{1 - n^2}{1 + n^2} = \frac{2\,m - m^2}{2 - 2\,m + m^2} = m + \tfrac{1}{2}m^2 + \&\text{c.} = m' \tag{758}$$

$$= .0033389$$

$$= m \text{ nearly}$$

$$\delta L = \frac{m'}{\sin.\,1''} \sin.\,(2\,L - \delta L) \tag{759}$$

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$$= \frac{m'}{\sin. 1''} \sin. 2\,L \text{ nearly}$$

$$= \frac{1}{300 \sin. 1''} \sin. 2\,L = \frac{1}{5 \sin. 1'} \sin. 2\,L \text{ nearly} \qquad (760)$$

$$= \frac{\sin. 2\,L}{\sin. 5'} \text{ nearly.}$$

139. *Problem. To find the parallax in latitude and longitude.*

Solution. Let Z (fig. 55) be the zenith, P the pole of the ecliptic, and M' the apparent place of the body whose parallax is sought, and M its true place. Let also

$B = PZ =$ the zenith distance of the pole,
$=$ the altitude of the nonagesimal,
$A = 90° - ZM' =$ the apparent altitude,
$A' = 90° - ZM =$ the true altitude,
$D = 90° - PM =$ the true latitude of the body,
$h = ZPM =$ the true diff. of long. of the body and the zenith,
$P =$ the horizontal parallax,
$p = P \cos. A = MM' =$ the parallax in altitude,
$\delta h = ZPM' - ZPM =$ the parallax in longitude,
$\delta D = PM' - PM =$ the parallax in latitude,
$D' = D - \delta D$.

The triangles PMM' and ZPM' give

$$\delta h = \frac{p \sin. M'}{\cos. D} = \frac{p \sin. B \sin. (h + \delta h)}{\cos. A \cos. D}$$

$$= P \sin. B \sec. D \sin. (h + \delta h). \qquad (761)$$

Draw PN to bisect the angle MPM', draw MH and $M'H'$ perpendicular to PN, join ZH and ZH', and we have

$$\delta D = HH' = HN + H'N'$$
$$= MN \cos. N + M'N \cos. N$$
$$= (MN + M'N) \cos. N = MM' \cos. N$$
$$= P \cos. A \cos. N. \qquad (762)$$

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But the triangle $ZH'M'$ gives, by putting

$$N' = ZH'H, \quad ZH' = 90° - A'',$$

since $$H'M'N = 90° - N$$

$$\cos. A'' : \cos. N = \cos. A : \cos. N';$$

whence $$\cos. A \cos. N = \cos. A'' \cos. N'$$

and $$\delta D = P \cos. A'' \cos. N'.$$

Produce $H'Z$ and $H'P$ to E and C,

making $$90° = H'E = H'C.$$

The right triangle ZEC will give

$$EC = N', \quad ZE = 90° - ZH' = A''$$

$$\cos. ZC = \cos. ZE \cos. EC = \cos. A'' \cos. N',$$

whence $$\delta D = P \cos. ZC; \tag{763}$$

and the triangle ZPC gives

$$PC = 90° - PH' = D' \text{ nearly},$$

$$ZPC = 180° - ZPH' = 180° - (h + \tfrac{1}{2}\delta h),$$

whence, by (307),

$$\cos. ZC = \cos. B \cos. D' - \sin. B \sin. D' \cos. (h + \tfrac{1}{2}\delta h)$$

$$\delta D = P \cos. B \cos. D' - P \sin. B \sin. D' \cos. (h + \tfrac{1}{2}\delta h), \tag{764}$$

and formulas (761) and (764) agree with the rule in the Navigator [B. p. 404].

140. *Corollary.* By putting

$$k = P \sin. B \sec. D, \tag{765}$$

(761) becomes

$$\delta h = k \sin. (h + \delta h)$$
$$= k \sin. h \cos. \delta h + k \cos. h \, \delta h.$$

Hence, if

$$n = k \cos. h \tag{766}$$

$$(1 - n)\, \delta h = k \sin. h \cos. \delta h$$

$$\delta h = \frac{k \sin. h}{(1-n) \sec. \delta h} = \frac{P \sin. B \sec. D \sin. h}{(1-n) \sec. \delta h}. \tag{767}$$

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The logarithm of the reciprocal of $1-n$ is called the correction for n, and is found from Table I, at the end of this volume, where it is placed opposite to the log. n.

141. *Corollary.* Another process for computing $\delta\, D$ may be obtained from (762). This equation gives

$$\begin{aligned}\delta\, D &= P \cos. N \cos. (A'-p)\\ &= P \cos. N \cos. A' \cos. p + P\, p \cos. N \sin. A'\\ &= P \cos. N \cos. A' \cos. p + P . P \cos. A \cos. N \sin. A'\\ &= P \cos. N \cos. A' \cos. p + P\, \delta\, D \sin. A'. \end{aligned} \tag{768}$$

Let
$$n' = P \sin. A', \tag{769}$$

and (768) gives

$$(1-n')\, \delta\, D = P \cos. N \cos. A' \cos. p$$

$$\delta\, D = \frac{P \cos. N \cos. A'}{(1-n') \sec. p}. \tag{770}$$

The triangle ZMH gives, by putting

$$N'' = ZHH', \;\; ZH = 90^\circ - A''',$$

since
$$HMZ = 90^\circ + N$$

$$\cos. A''' : \cos. N = \cos. A' : \cos. N'';$$

whence
$$\cos. A''' \cos. N'' = \cos. N \cos. A'$$

$$\delta\, D = \frac{P \cos. A''' \cos. N''}{(1-n') \sec. p}, \tag{771}$$

and A''' and N'' can be deduced by direct solution of the triangle ZHP, in which

$$ZPH = h + \tfrac{1}{2}\, \delta\, h, \;\; PH = PM = 90^\circ - D \text{ nearly},$$

and A''' may be substituted for A' in determining the value of the small quantity n' by means of (769), and sec. $\delta\, D$ may be substituted for sec. p.

142. *Problem. To find the parallax in right ascension and declination.*

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Solution. Formulas (761–771) may be applied immediately to this case, by putting

$B =$ the altitude of the equator $=$ the co-latitude,

$D =$ the true declination,

$D' =$ the apparent declination,

$h =$ the right ascension of the body diminished by that of the zenith $=$ the hour angle of the body,

$\delta D =$ the parallax in declination,

$\delta h =$ the parallax in right ascension.

And formulas (761, 767, 771) correspond to those given by Woodhouse, in his method of calculating eclipses and occultations, in the Nautical Almanac for 1826. The mean values of sec. δD and sec. p are there substituted for them, which is 0.00006.

143. The *apparent diameter* of a heavenly body is the angle which its disc subtends.

144. *Problem. To find the apparent semidiameter of a heavenly body.*

Solution. Let O' (fig. 56) be the centre of the heavenly body, A the observer, and AT the tangent to the disc of the body. The angle TAO' is the apparent semidiameter. Let

$$R_1 = O'T$$

$$\sigma = O'AT.$$

$$r = AO',$$

we have

$$\sin.\ \sigma = \frac{O'T}{AO'} = \frac{R_1}{r}. \tag{772}$$

Hence, by (fig. 53), if A is the apparent altitude of the body, A' the true altitude,

$$\sin.\ \sigma = \frac{R_1 \sin.\ p}{R \cos.\ (A+p)} = \frac{R_1 \sin.\ p}{R \cos.\ A'} \tag{773}$$

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$$\sigma = \frac{R_1}{R} p \text{ sec. } (A + p)$$

$$= \frac{R_1 P}{R} \quad \frac{\text{cos. } A}{\text{cos. } A'}. \tag{774}$$

But if Σ is the horizontal semidiameter, we have

$$\Sigma = \frac{R_1 P}{R} \tag{775}$$

which is also the semidiameter, as seen from the earth's centre; whence (774) becomes

$$\sigma = \Sigma \frac{\text{cos. } A}{\text{cos. } A'}, = \Sigma \frac{\text{cos. } (A' - p)}{\text{cos. } A'}$$

$$= \Sigma \frac{\text{cos. } A' + p \text{ sin. } A'}{\text{cos. } A'} = \Sigma (1 + P \text{ sin. } A'), \tag{776}$$

or, by (769),

$$\sigma = \Sigma (1 + n) \tag{777}$$

$$= \frac{\Sigma}{1 - n} \text{ (nearly)} \tag{778}$$

$$= P \frac{R_1}{R} \cdot \frac{1}{1 - n}.$$

145. *Corollary.* We have

$$R_1 = 0.2725 \; R \tag{779}$$

$$R = 3.67 \quad R_1 \tag{780}$$

whence $\log. \frac{R_1}{R} = 9.43537$, (ar. co.) $= 0.5646$, (781)

so that formula (775) agrees with [B. p. 443. No. 10 of the Rule].

146. *Corollary.* If $\delta \sigma$ is the augmentation of the semidiameter for the altitude A, we have, by (776),

$$\delta \sigma = \Sigma P \text{ sin. } A' = \Sigma P \text{ sin. } A$$

$$= \frac{R_1}{R} P^2 \text{ sin. } A \tag{782}$$

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or, in order to express $\delta\,\sigma$ and P in seconds,

$$\delta\,\sigma = \frac{R_1}{R}\,P^2 \sin.\,1'' \sin.\,A. \qquad (783)$$

Now for the mean horizontal parallax of 57′ 30″, we have

$$\log.\,\frac{R_1}{R}\,P^2 \sin.\,1'' = 1.19658 \qquad (784)$$

$$\frac{R_1}{R}\,P^2 \sin.\,1'' = 15.72, \qquad (785)$$

agreeing very nearly with the explanation to Table XV of the Navigator.

147. *Corollary.* The augmentation can also be calculated without determining the altitude. Thus, from (774)

$$\delta\,\sigma = \Sigma\left(\frac{\cos.\,A}{\cos.\,A'} - 1\right). \qquad (786)$$

But from (fig. 55) and (761)

$$\cos.\,A = \sin.\,ZM' = \frac{\sin.\,(h + \delta\,h)\,.\,\cos.\,(D - \delta\,D)}{\sin.\,Z} \qquad (787)$$

$$\cos.\,A' = \sin.\,ZM = \frac{\sin.\,h\,\cos.\,D}{\sin.\,Z} \qquad (788)$$

$$\frac{\cos.\,A}{\cos.\,A'} - 1 = \frac{\sin.\,(h + \delta\,h)\,.\,\cos.\,(D - \delta\,D)}{\sin.\,h\,\cos.\,D} - 1 \qquad (789)$$

$$= \frac{\cos.\,h\,\cos.\,(D - \delta\,D)\,\delta\,h}{\cos.\,D\,\sin.\,h} + \frac{\cos.\,(D - \delta\,D)}{\cos.\,D} - 1$$

$$= \frac{P\,.\cos.\,h\,.\sin.\,B\sin.\,(h + \delta\,h)}{\cos.\,D\sin.\,h} + \frac{\cos.\,(D - \delta\,D)}{\cos.\,D} - 1$$

Now the latitude of the moon is so small, that, in the first term, we may put

$$\cos D = 1, \qquad (790)$$

which gives by (786), and putting

$$H = \Sigma\,P\,.\,\cos.\,h\,\sin.\,B \qquad (791)$$

$$H' = \Sigma\left(\frac{\cos.\,(D - \delta\,D)}{\cos.\,D} - 1\right) \qquad (792)$$

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$$\begin{aligned}\delta\,\sigma &= H + H \cot.\, h\; \delta\, h + H' \\ &= H + H \,.\, P \,.\, \cos.\, h \sin.\, B + H' \\ &= H + \frac{H^2}{\Sigma} + H'. \end{aligned} \tag{793}$$

Now we have, by (791) and (792),

$$H = \tfrac{1}{2}\, \Sigma \,.\, P \,.\, [\sin.\, (B + h) + \sin.\, (B - h)] \tag{794}$$

$$H' = \Sigma\, (\text{tang.}\, D \,.\, \delta\, D + \cos.\, \delta\, D - 1), \tag{795}$$

and formulas (793 to 795) agree with the method of calculating the augmentation of the semidiameter given in Table XLIV of the Navigator. The three first parts of this table are calculated for the value of Σ,

$$\Sigma = 16' = 960''$$

whence $$\tfrac{1}{2}\, \Sigma \,.\, P = 8''.18.$$

The fourth part of the table is the correction which arises from the difference between the actual value of Σ and that assumed in the three former parts. If we put

$$\delta'\, \sigma = \text{the value of } \delta\, \sigma \text{ for } \Sigma = 16',$$

we have, by (782) and (795),

$$\delta\, \sigma : \delta'\, \sigma = \Sigma^2 : (16')^2 \tag{796}$$

$$\begin{aligned}\delta\, \sigma &= \frac{\Sigma^2}{256}\, \delta'\, \sigma \\ &= \delta'\, \sigma + \left(\frac{\Sigma^2}{256} - 1\right) \delta'\, \sigma \\ &= \delta'\, \sigma + \frac{\Sigma^2 - 256}{256}\, \delta'\, \sigma \\ &= \delta'\, \sigma + \frac{(\Sigma + 16)\,(\Sigma - 16)}{256}\, \delta'\, \sigma, \end{aligned} \tag{797}$$

as in the explanation of this table.

148. Examples.

1. Find a planet's parallax in altitude, when its horizontal parallax is 25″, and its altitude 30°.

Ans. 22″.

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2. Calculate the reduction of parallax for parallax 61′, and latitude 82°.

Solution. We have in (752), $\frac{1}{10} P = 6.1$

$2\,L = 164°$, $\cos. 2\,L = -.961$, $1 - \cos. 2\,L = 1.961$

$$\delta p = 12''.0$$

3. Calculate the reduction of parallax for parallax 57′, and latitude 22°.

Ans. 1″.6.

4. Calculate the reduction of parallax for parallax 53′, and latitude 68°.

Ans. 7″.9.

5. Calculate the reduction of latitude for latitude 70°.

Solution. We have by (759)

	$\frac{m'}{\sin. 1''}$		cos.	2.83804
$2\,L =$	$140°$		sin.	9.80807
1st app. $\delta L =$	$0°\ 7'\ 23''$	$=$	$443''$	2.64611
$2\,L - \delta L =$	$139°\ 52'\ 37''$		sin.	9.80918
$\delta L =$	$7'\ 23''.8$	$=$	$443''.8$	2.64722

6. Calculate the reduction of latitude for latitude 20°.

Ans. 7′ 21″.5.

7. Calculate the reduction of latitude for latitude 50°.

Ans. 11′ 18″.6.

8. Find the moon's parallax in latitude and longitude, when her horizontal parallax is 59′ 10″.3; her latitude 3° 7′ 19″ S., her longitude 44° 36′ 16″; the altitude of the nonagesimal 37° 56′ 14″, its longitude 25° 27′ 16″, the latitude of the place 43° 17′ 18″ N.

Solution. By (761) and (764),

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Reduced parallax $= 59' \, 10''.3 - 5''.3 = 59' \, 5'' = 3545''$

Reduced latitude $= 43° \, 17' \, 18'' - 11' \, 27'' = 43° \, 5' \, 51''$

$$h = 44° \, 36' \, 16'' - 25° \, 27' \, 16'' = 19° \, 9'$$

3545		3.54962			3.54962		3.550
37° 56′ 14″	sin.	9.78873		cos.	9.89691	sin.	9.789
3° 7′ 19″	sec.	0.00064	3° 7′ 19″	cos.	9.99936		
		3.33899	46′ 32″		3.44589		
19° 9′	sin.	9.51593	3° 53′ 51″	cos.	9.99899		
12′		2.85492	46′ 30″		3.44552		
19° 21′	sin.	9.52027	3° 53′ 49″			sin.	8.831
$\delta h = 12' \, 3''$		2.85926		19° 15′		cos.	9.975
19° 21′ 3″			—2′ 20″				2.145
			$\delta D = 44' \, 10''$				

9. Find the moon's parallax in latitude and longitude, when the horizontal parallax is 60′ 6″.2; her latitude 1° 30′ 12″ N., her longitude 130° 17′, the altitude of the nonagesimal 85° 14′, its longitude 125° 17′, the latitude of the place 46° 11′ 28″.4 N.

Ans. Parallax in longitude = 5′ 18″

Parallax in latitude = 4′ 30″.5.

10. Calculate the parts of Table XLIV, when the argument of the first part is $3^s \, 19° = 109°$; that of the second 12″.4, the moon's true latitude 1° 20′ N., the moon's parallax in latitude 50′, the sum of the three first parts 13″, and the moon's horizontal semidiameter 14′ 50″.

Solution. $8''.1845 \text{ sin. } 109° = 7''.74 = \text{Part I.}$

$$\text{Part II} = \frac{(12''.4)^2}{960''} = 0''.16.$$

$$\begin{aligned} \text{Part III} &= 960'' \, [\text{sin. } 50' \text{ tang. } 1° \, 20' - 1 + \text{cos. } 50'] \\ &= 960'' \, [\text{sin. } 50' \text{ tang. } 1° \, 20' - 2 \text{ sin.}^2 \, 25'] \\ &= 960'' \, [0.00023] = 0''.22. \end{aligned}$$

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$$\text{Part IV} = -13'' \times \frac{30'\,50'' \times 1'\,10''}{256'} = -\frac{13'' \times 30.83 \times 1.17}{256}$$

$$= -1''.83.$$

11. Calculate the parts of Table XLIV, when the argument of the first part is $2^s\ 16°$, that of the second $15''.5$, the moon's true latitude $3°$ S., the moon's parallax in latitude $30'$, the sum of the three first parts $11''$, and the moon's horizontal semidiameter $15'\ 20''$.

Ans. Part I $= 7''.94$
Part II $= 0\ .25$
Part III $= -0\ .48$
Part IV $= -0\ .90$

12. Calculate the number of Table XV, when the altitude is $45°$.

Ans. $11''$.

13. Calculate the augmentation of the moon's semidiameter in Example 8; when the horizontal semidiameter is $16'\ 50''$.

Solution.	Part I $=$	$6''.87 + 2''.58$	$=$	$9''.45$
	Part II $=$			$0\ .09$
	Part III $=$		$-$	$0\ .75$
		sum	$=$	$8''.79$
	Part IV $=$			$0\ .92$
		augmentation	$=$	$9''.71$

14. Calculate the augmentation of the moon's semidiameter, in Example 9, when the horizontal semidiameter is $15'\ 30''$.

Ans. $15''.54$.

15. Calculate the moon's parallax in right ascension and declination, and her augmented semidiameter, for the Cambridge Observatory, when her hour angle is $57°\ 46'\ 48''$, declination $21°\ 42'\ 55''$ S., and horizontal parallax $61'\ 16''.9$.

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Solution.

$P =$ the reduced parallax $= 61' \, 16''.9 - 5''.6 = 61' \, 11''.3 = 3671''.3$

$90° - B =$ reduced latitude $= 42° \, 22' \, 48'' - 11' \, 26'' = 42° \, 11' \, 22''$

P	3.56482	$h =$	57° 46′ 48″		
sin. B	9.86978	$\frac{1}{2} \delta h$	20 48	tan. B.	0.04268
sec. D	0.03197	$h + \frac{1}{2} \delta h$	58° 7′ 36″	cos.	9.72268
k	3.46657		3.46657	tan. θ	9.76536
h cos.	9.72687	sin.	9.92737	$\theta =$	30° 13′ 30″
n	3.19344	corr.	330	$D = -$	21° 42′ 55″
sec. δh			3	$\theta' =$	8° 30′ 35″
$\delta h = 2495''.8$			3.39721	sin. θ	9.70191
θ'		tan.	9.17500	sec.	0.00481
				tan. $(h + \frac{1}{2} \delta h)$	0.20635
N''		cos.	9.88863	tan.	9.91307
A'''		tan.	9.06363	sin.	9.06074
A'''		cos.	9.99711	P	3.56482
		P	3.56482	n'	2.62556
		n' corr.	89		89
		sec. δD	6	hor. par.	3.36548
$\delta D = 2827''.4$			3.45139	const.	9.43537
				$s = 1004''.0$	3.00179

16. Calculate the moon's parallax in right ascension and declination, and her augmented semidiameter, for Providence, when her hour angle is 58° 0′ 18″, declination 21° 42′ 52″ S., and horizontal parallax 61′ 16″.2.

The latitude of Providence is 41° 49′ 22″ N.

Ans. The parallax in right ascension = 2523″.2
" " declination = 2803″.9
the augmented semidiameter = 1003″.8

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17. Calculate the moon's parallax in right ascension and declination, and her augmented semidiameter, for Mount Joy Observatory, Portland, when her hour angle is $58° 15' 54''$, declination, $21° 42' 52''$ S., and horizontal parallax $61' 16''.2$.

The latitude of Mount Joy Observatory is $43° 39' 52''$ N.

Ans. The parallax in right ascension $= 2426''.0$
" " declination $= 2864''.2$
the augmented semidiameter $= 1003''.8$

18. Calculate the moon's parallax in right ascension and declination, and her augmented semidiameter, for Mr. Bond's observatory, in Dorchester, when her hour angle is $60° 38' 34''$, declination $22° 42' 8''$ N., and horizontal parallax $56' 14''.4$.

The latitude of Mr. Bond's observatory is $42° 19' 10''$.

Ans. The parallax in right ascension $= 2375''.3$
" " declination $= 1632''.9$
the augmented semidiameter $= 928''.5$

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CHAPTER XII.

ECLIPSES.

149. A SOLAR *eclipse* is an obscuration of the sun, arising from the moon's coming between the sun and the earth; and occurs therefore at the time of new moon.

It is central to an observer, when the centre of the moon passes over the sun's centre. It is *total*, when the moon's apparent disc is larger than the sun's, and totally hides the sun. It is annular, when the moon's apparent disc is smaller than the sun's, but is wholly projected upon the sun's disc.

The *phase* of an eclipse is its state as to magnitude.

150. An *occultation of a star or planet* is an eclipse of this star or planet by the moon.

A *transit* of Venus or Mercury is an eclipse of the sun by one of these planets.

151. *Problem. To find when a solar eclipse will take place.*

Solution. Let O (fig. 57) be the sun's centre, and O_1 the moon's centre at the time of new moon, and let

$$\begin{aligned}\beta &= \text{the latitude of the moon at new moon}\\ &= OO_1.\end{aligned}$$

Let ON be the ecliptic, and N the moon's node, so that NO_1 is the moon's path. Let

$$N = \text{the inclination of the moon's orbit to the ecliptic};$$

Draw OP perpendicular to the moon's orbit, and if, when the moon

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arrives at P, the sun arrives at O', the least distance of the centres of sun and moon is nearly equal to $O'P$. Now the triangle OPO_1 gives

$$OP = \beta \cos. N = \beta - \beta (1 - \cos. N)$$
$$= \beta - 2 \beta \sin.^2 \tfrac{1}{2} N = \beta - \tfrac{1}{2} \beta \sin.^2 N$$

$n =$ ratio of the sun's mean motion divided by the moon's

$$= \tfrac{1}{12} \text{ nearly,} \qquad (798)$$

we have $$OO' = n \times O_1P = n \beta \sin. N.$$

Draw $O'B$ perpendicular to OP, and we have nearly

$$OB = OP - O'P = OO' \sin. N$$
$$= n \beta \sin.^2 N.$$

Hence

$$O'P = \beta - (\tfrac{1}{2} + n) \beta \sin.^2 N = \beta - \tfrac{7}{12} \beta \sin.^2 N. \qquad (799)$$

The apparent distance of the centres of the sun and moon is affected by parallax, and the true distance is diminished as much as possible for that observer, who sees the sun and moon in the horizon, and OP vertical, in which case the diminution is equal to the difference of the horizontal parallaxes of the sun and moon. Let, then,

$\pi =$ the moon's horizontal parallax,

$\Pi =$ the sun's horizontal parallax,

$\Delta =$ the apparent distance of the centres,

we have

$$\text{the least apparent dist.} = OP - (\pi - \Pi)$$
$$= \beta - \tfrac{7}{12} \beta \sin.^2 N - \pi + \Pi. \qquad (800)$$

Now, an eclipse will take place, when this least apparent distance of the centres is less than the sum of the semidiameters of the sun and moon. Thus, let

$s =$ the moon's semidiameter,

$\sigma =$ the sun's semidiameter.

In case of an eclipse, we must have

$$\beta - \tfrac{7}{12} \beta \sin.^2 N - \pi + \Pi < s + \sigma, \qquad (801)$$

or $$\beta < \pi - \Pi + s + \sigma + \tfrac{7}{12} \beta \sin.^2 N. \qquad (802)$$

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152. *Corollary.* We have, by observation,

the greatest value of π	$=$	$61' \; 32''$,
the least value	$=$	$52' \; 50''$,
the mean value	$=$	$57' \; 11''$,
the greatest value of Π	$=$	$9''$,
the least value	$=$	$8''$,
the greatest value of s	$=$	$16' \; 46''$,
the least value	$=$	$14' \; 24''$,
the mean value	$=$	$15' \; 35''$,
the greatest value of σ	$=$	$16' \; 18''$,
the least value	$=$	$15' \; 45''$,
the mean value	$=$	$16' \; 1''$,
the greatest value of N	$=$	$5° 20' \; 6''$,
the least value	$=$	$4° 57' \; 22''$,
the mean value	$=$	$5° \; 8' \; 44''$.

Now, in the last term of (802) we may put for N its mean value, and for β its mean value obtained by supposing it equal to the preceding terms, which gives

$$\beta = \pi - \Pi + s + \sigma = 88' \; 38'' = 5318'' \tag{803}$$

$$\tfrac{7}{12} \beta = 3102''$$

$$\sin. N = \sin. 5° \; 8' \; 44'' = 0.09, \; \sin.^2 N = 0.008$$

$$\tfrac{7}{12} \beta \sin.^2 N = 25'', \tag{804}$$

whence (802) becomes

$$\beta < \pi - \Pi + s + \sigma + 25''. \tag{805}$$

153. *Corollary.* If, in (805), the greatest values of π, s, and σ, and the least value of Π are substituted, the limit

$$\beta < 1° \; 34' \; 52''$$

is the greatest limit of the moon's latitude at the time of new moon, for which an eclipse can occur.

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154. *Corollary.* If, in (805), the least values of π, s, and σ, and the greatest values of Π are substituted, the limit

$$\beta < 1^\circ\ 23'\ 15''$$

is the least limit of the moon's latitude at the time of new moon, for which an eclipse can fail to occur.

155. *Problem. To find when a lunar eclipse will happen.*

Solution. The solution is the same as in § 151, except that the semidiameter of the earth's shadow at the distance of the moon is to be substituted for that of the sun; and the change in the position and apparent magnitude of the moon from parallax may be neglected, because when the earth's shadow falls upon the moon, the moon is eclipsed to all who can see it. Now if S (fig. 63) is the sun, E the earth, GF the semidiameter of the ~~sun's~~ earth's shadow at the moon, we have

$$\begin{aligned}\text{the app. semi.} &= FEG = EFL - EIF = \pi - EIF \\ &= \pi - (KES - EKI) \\ &= \pi - \sigma + \Pi,\end{aligned}$$

or rather, this would be the apparent semidiameter, if it were not for the earth's atmosphere, which increases the breadth of the shadow about $\frac{1}{60}$th part; so that

$$\text{the app. semidiam.} = \tfrac{61}{60}\,(\pi - \sigma + \Pi),$$

and therefore, in order that an eclipse must happen, we must have, by (802),

$$\beta = \text{the latitude at the time of full moon,}$$
$$\beta < \tfrac{61}{60}\,(\pi + \Pi - \sigma) + s + \tfrac{7}{12}\,\beta \sin.^2 N. \qquad (806)$$

156. *Corollary.* In the last term of (806), we may put for N its mean value, and for β its mean value obtained by supposing it equal to the preceding terms, which gives

$$\beta = 57'\ 35'' = 3455'',\ \tfrac{7}{12}\,\beta = 2015''$$
$$\sin.^2 N = 0.008,\ \tfrac{7}{12}\,\beta \sin.^2 N = 16'',$$

whence (806) becomes

$$\beta < \tfrac{61}{60}\,(\pi + \Pi - \sigma) + s + 16''. \qquad (807)$$

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157. *Corollary.* If, in (807), the greatest values of π, Π, and s are substituted, and the least value of σ, the limit

$$\beta < 63' \, 45''$$

is the greatest limit of the moon's latitude at the time of full moon, for which an eclipse can occur.

158. *Corollary.* If, in (807), the least values of π, Π, and s are substituted, and the greatest value of σ, the limit

$$\beta < 51' \, 57''$$

is the least limit at which an eclipse can fail to occur.

159. ***Problem.*** ***To calculate when a given phase of a lunar eclipse will occur.***

Solution. If in (fig. ~~17~~ 57) NPO_1 is the path of the moon relatively to the centre of the earth's shadow which is at O, the required computation consists, simply, in finding the instant when the moon's distance from O is that which corresponds to the required phase. The indefiniteness of the outline of the earth's shadow renders an accurate calculation superfluous, and it is sufficient to regard $O_1 ON$ as a plane triangle.

160. *Corollary.* At the beginning or end of the lunar eclipse, we have

$$\begin{aligned} \Delta &= \text{the distance of the centres of the moon and shadow} \\ &= \tfrac{61}{60}(\pi + \Pi - \sigma) \pm s, \qquad (808) \end{aligned}$$

in which the upper sign corresponds to the first and last contacts with the shadow, and the lower sign to the beginning and end of the total phase.

161. ***Problem.*** ***To compute the general circumstances of a solar eclipse.***

Solution. This problem will be found to subdivide itself naturally into several others, but the general mode of solution may be developed in a preliminary view of the whole question. The method here given is substantially *Bessel's.*

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The moon's shadow upon the earth is the geometrical intersection of a right cone, which is in contact with the sun and moon, and for every point within this shadow there is a *total* eclipse of the sun. If, however, this shadow does not reach the earth, there will still be within the limits of the *umbral* cone produced beyond its vertex, an eclipse of a portion of the sun equal to the apparent size of the moon, and this dark portion, surrounded by the bright ring of the uneclipsed portion of the sun, constitutes an *annular* eclipse. But there is also an eclipse beyond the limits of this cone of all that portion of the sun which is hidden by the moon, and, therefore, for every place included within the *penumbral* cone which is drawn in contact with the sun and moon, and which has its vertex between these two bodies; but this is a *partial* eclipse.

A plane may now be supposed to be drawn through the earth's centre, perpendicular to the line which joins the centres of the sun and moon. The moon's shadow and penumbra upon this plane are concentric circles, and the path of their common centre upon this plane may be computed and described. Any point of the earth may be referred to this plane by a line drawn from the vertex of the cone through the point, and the relative position of the common intersection of this line with the plane and the moving shadow or penumbra of the moon, will show the successive phases of an eclipse at that point.

It will promote perspicuity to carve the problem into several subdivisions.

162. *Problem. To find the position of the line which is drawn through the earth's centre parallel to the line joining the centres of the sun and moon.*

Solution. In the triangle formed by joining the centres of the sun, moon, and earth, the angle at the earth is the apparent angular distance of the sun and moon, and the angle at the sun is the angle which the required line makes with the line drawn to the sun.

Let $\gamma =$ the angular distance of the sun and moon,

$c =$ the angle at the sun;

$r' =$ the distance of the moon from the earth,

$r =$ the distance of the sun;

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we have

$$\sin. c = \frac{r'}{r} \sin. \gamma, \tag{809}$$

or on account of the smallness of c

$$c = \frac{r'}{r} \gamma. \tag{810}$$

Since the line which is drawn from the earth's centre parallel to that which joins the centres of the sun and moon is in the plane of the above triangle, it cuts the surface of the celestial sphere at a point (F) which is in the arc of the great circle joining the sun and moon, and produced on the side of the sun by a distance equal to c.

163. *Corollary.* By putting

$\Pi =$ the sun's equatorial horizontal parallax

$\pi =$ the moon's equatorial horizontal parallax

$\Pi' =$ the mean value of $\Pi = 8''.5776$

$m = \frac{r'}{r}$

we have the following form in the computation of m,

$$m = \frac{\sin. \Pi}{\sin. \pi} = \frac{\sin. \Pi'}{r \sin. \pi}$$

$$\begin{aligned} \log. m &= \log. \sin. \Pi' - \log. r - \log. \sin. \pi \\ &= 5.6189 \qquad - \log. r - \log. \sin. \pi \end{aligned} \tag{811}$$

in which r is expressed in unity of the sun's mean distance.

164. *Corollary.* The right ascension and declination of the point (F) may easily be computed from the sun's right ascension and declination. Let

$\alpha =$ the sun's right ascension

$a =$ the right ascension of F

$\delta =$ the sun's declination

$d =$ the declination of F

$l =$ the sun's longitude

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λ = the (moon's — sun's) longitude

β = the (moon's — sun's) latitude

O = the obliquity of the ecliptic

u = the angle which γ makes with the circle of latitude drawn through the sun

ω = the angle which γ makes with the circle of declination drawn through the sun.

If then (fig. 59), S is the sun's place in the ecliptic, M the moon's relative place, N the pole of the ecliptic, Z that of the equator, we have

$$MSN = u, \qquad MSZ = \omega,$$

$$MS = \gamma \qquad ZN = O,$$

$$\text{tan. } u = \text{sin. } \lambda \text{ cot. } \beta \tag{812}$$

$$\text{tan. } \gamma = \text{tan. } \lambda \text{ cosec. } u, \tag{813}$$

from which u and γ may be computed, and the substitution of (813) in (809) gives by (811)

$$c = m \text{ tan. } \gamma \text{ cos. } \gamma \text{ cosec. } 1''. \tag{814}$$

The sun's place in the ecliptic gives

$$\text{cosin. } l = \text{cot. } O \,.\, \overset{\text{tan.}}{\cancel{\text{cot.}}}\ ZSN = \text{cot. } O \text{ tan. } (u - \omega)$$

$$\underset{\text{tan}}{\cancel{\text{cot.}}}\ (u - \omega) = \text{cos. } l \text{ tan. } O, \tag{815}$$

from which $u - \omega$ may be computed, and thence ω. We then have obviously

$$\delta - d = c \text{ cos. } \omega, \tag{816}$$

$$\alpha - a = c \text{ sin. } \omega \text{ sec. } \delta. \tag{817}$$

165. *Problem. To find the path of the centre of the moon's shadow upon the plane which passes through the earth's centre perpendicular to the line which joins the centres of the sun and moon.*

Solution. The angle which the line drawn from the moon to the earth makes with that drawn to the centre of the shadow, which is simply the continuation of the line drawn from the sun, is $c + \gamma$.

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Hence if ϱ is the distance from the earth's centre to the centre of the shadow, we have

$$\varrho = r' \sin. (c + \gamma) = \frac{1}{\sin. \pi} \sin. (c + \gamma), \tag{818}$$

in which r' and ϱ are expressed in units of the earth's equatorial radius.

The direction of the line ϱ may be conveniently referred to the intersection of the plane of reference with the circle of declination drawn through the earth's centre and the point F. Let

ω' = the angle which ϱ makes with the line of intersection,

and ω' is evidently the inclination of the arc c to the circle of declination drawn through F. It differs, therefore, very little from ω, and the difference may be found from the triangle formed with c, and the circles of declination passing through F and the sun to be

$$\omega - \omega' = c \sin. \omega \tan. \delta. \tag{819}$$

166. *Corollary.* Let x be the distance of the centre of the shadow from the above line of intersection, and y the elevation towards the north of the foot of the perpendicular let fall from the centre of the shadow upon this line of intersection above the earth's centre, and we have

$$x = \varrho \sin. \omega', \tag{820}$$

$$y = \varrho \cos. \omega'. \tag{821}$$

167. *Problem. To find the umbral and penumbral radii upon the plane of reference of the preceding problem.*

Solution. Either of these radii is plainly equal to the product of the distance of the vertex of the cone from the plane, by the tangent of the angle of the cone. If then

H = the apparent semidiameter of the sun at his mean distance
= 959″.788

K = the ratio of the moon's radius divided by that of the earth
= 0.27227

f = the angle of the cone

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$S =$ the distance of the vertex of the cone from the plane of reference

$s' =$ the distance of the centres of the sun and moon

$= r \cos. c - r' \cos. (c + \gamma)$,

or since c is very small,

$$s' = r - r' \cos. (c + \gamma)$$

$$= r [1 - m \cos. (c + \gamma)] \qquad (822)$$

The sun's radius $= \sin. H$

The earth's radius $= \sin. \Pi$

The moon's radius $= K \sin. \Pi$

$$\sin. f = \frac{\sin. H \mp K \sin. \Pi}{s'}, \qquad (823)$$

in which the upper sign corresponds to the umbral and the lower to the penumbral cone, and s' is expressed in units of the sun's mean distance; we have, moreover, by taking the earth's equatorial radius as the unit

$z =$ the moon's distance from the plane of reference

$$= r' \cos. (c + \gamma) = \frac{1}{\sin. \pi} . \cos. (c + \gamma) \qquad (824)$$

$$S = z \mp \frac{K}{\sin. f} \qquad (825)$$

$\rho' =$ the radius of the shadow

$= S \tan. f$

$$= z \tan. f \pm K \sec. f. \qquad (826)$$

168. *Corollary.* We have in (823)

$$\log. (\sin. H - K \sin. \Pi) = 7.66669 \qquad (827)$$

$$\log. (\sin. H + K \sin. \Pi) = 7.66880. \qquad (828)$$

169. *Corollary.* For any plane which is drawn parallel to the plane of reference, and at a distance z' from it towards the vertex of the cone, the radius of the shadow will be diminished by

$$z' \tan. f, \qquad (829)$$

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and the relative position of the centre of the shadow and of the point of intersection with the line drawn through the centre of the earth parallel to the axis of the cone will remain unchanged.

170. *Problem. To find the position of any point of the earth's surface with reference to the axis of the shadow.*

Solution. Let (fig. 35) $NESW$ represent the plane of reference drawn through the centre of the earth, P the north pole, Z the point in which the line drawn from the earth's centre parallel to the axis of the cone cuts the surface, and B the place. Let

$$\theta' = \text{the reduced latitude of the place,}$$
$$\lambda' = \text{its longitude,}$$
$$R = \text{its distance from the centre,}$$
$$\mu' = BPC;$$
$$d = PN,$$

and let now that plane of reference be adopted which is drawn through B parallel to the original plane. The line of intersection of this plane with the plane of the meridian NZS corresponds to the line NS in the original plane. If BC is drawn perpendicular to NZS, we have

$$x' = \text{the distance of } B \text{ from this line}$$
$$= R \sin. BC$$
$$= R \cos. \theta' \sin. \mu' \quad (830)$$

y' = the distance of the foot of the perpendicular from B upon this line from the intersection of the plane with the line from the centre to Z

$$= R \cos. BN = R \sin. \theta' \cos. d - R \cos. \theta' \sin. d \cos. \mu' \quad (831)$$

$$z' = \text{the height of } B \text{ above the original plane}$$
$$= R \cos. BZ = R \sin. \theta' \sin. d + R \cos. \theta' \cos. d \cos. \mu'. \quad (832)$$

171. *Corollary.* The radius of the shadow or penumbra for this plane is

$$\varrho' \pm z' \tan. f, \quad (833)$$

the upper sign being for the shadow in a total eclipse, and the lower for the other cases.

172. *Corollary.* The distance Δ of the place B from the axis of the shadow is obviously given by the equation

$$\Delta^2 = (x - x')^2 + (y - y')^2. \tag{834}$$

173. *Problem. To investigate the condition of the commencement or termination of an eclipse.*

Solution. At either of these phases of an eclipse, the distance Δ is exactly equal to the radius of the shadow, or by (833 and 834)

$$(\varrho' \pm z' \tan. f)^2 = (x - x')^2 + (y - y')^2, \tag{835}$$

or by transposition

$$(x - x')^2 = (\varrho' \pm z' \tan. f)^2 - (y - y')^2. \tag{836}$$

The second member of this equation, being the difference of two squares, may be separated into the two factors

$$B' = (\varrho' \pm z' \tan. f) + (y - y') \tag{837}$$

$$C' = (\varrho' \pm z' \tan. f) - (y - y'), \tag{838}$$

or by (831 and 832)

$$\begin{aligned} B' &= \varrho' + y - R \sin. \theta' (\cos. d \mp \sin. d \tan. f) \\ &\quad + R \cos. \theta' \cos. \mu' (\sin. d \pm \cos. d \tan. f) \end{aligned} \tag{839}$$

$$\begin{aligned} C' &= \varrho' - y + R \sin. \theta' (\cos. d \pm \sin. d \tan. f) \\ &\quad - R \cos. \theta' \cos. \mu' (\sin. d \mp \cos. d \tan. f). \end{aligned} \tag{840}$$

Hence, if we put

$$B = \varrho' + y \tag{841}$$

$$C = - \varrho' + y \tag{842}$$

$$E = \cos. d + \sin. d \tan. f = \cos. (d - f) \sec. f \tag{843}$$

$$F = \cos. d - \sin. d \tan. f = \cos. (d + f) \sec. f \tag{844}$$

$$G = \sin. d - \cos. d \tan. f = \sin. (d - f) \sec. f \tag{845}$$

$$H = \sin. d + \cos. d \tan. f = \sin. (d + f) \sec. f \tag{846}$$

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we have, by a modification of Bessel's formulæ suggested by T. Henry Safford, Jr., for the penumbra

$$B' = \quad B - ER \sin. \theta' + GR \cos. \theta' \cos. \mu' \tag{847}$$

$$C' = -C + FR \sin. \theta' - HR \cos. \theta' \cos. \mu' \tag{848}$$

$$A'^2 = (x - x')^2 = B'C'. \tag{849}$$

The formulæ for the shadow in a total eclipse are obtained from (847 and 848) by interchanging E with F and G with H, or by making ϱ' negative in this case the formulæ may remain unchanged.

The values of x, B, C, E, F, G and H, which are independent of the place, may be computed for various times and arranged in a tabular form. The value of R may be found from (750).

174. *Corollary.* The value of μ' for different places changes with the longitude, so that if

$$\mu = \text{the value of } \mu' \text{ for the first meridian}$$
$$= \text{R. A. of the first meridian} - a,$$

we have

$$\mu' = \mu + \lambda, \tag{850}$$

and the value of μ for different times may be given in the table.

175. *Problem. To find the time of the beginning or ending of an eclipse at any place.*

Solution. If for any time the value of $B'C'$ is found to differ but little from A'^2, the instant of the required phase may be computed by the following process of approximation. Let for the assumed time

$$H'^2 = B'C'.$$

Let also

$$x'' = \text{the change of } x \text{ in one second}$$
$$y'' = \text{that of } y$$
$$\mu'' = \text{that of } \mu.$$

The changes of ϱ', E, F, G and H' are so small, that they may be

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neglected. Hence if B'', C'', A'' and H'' are the changes in one second of B', C', A' and H', we have very nearly

$$B'' = \quad y'' - GR \cos.\, \theta' \sin.\, \mu' \,.\, \sin.\, \mu'' \tag{851}$$

$$C'' = -y'' + HR \cos.\, \theta' \sin.\, \mu' \,.\, \sin.\, \mu'' \tag{852}$$

$$2\, H' H'' = B' C'' + C' B'' = (C' - B')\, y'' - (GC' - HB')\, R \cos.\, \theta' \sin.\, \mu' \sin.\, \mu'' \tag{853}$$

$$A'' = x'' - R \cos.\, \theta' \cos.\, \mu' \sin.\, \mu''. \tag{854}$$

Owing to the smallness of tan. f we may by (845) and (846) put in (853)

$$G = H = \sin.\, d, \tag{855}$$

whence (853) becomes

$$2\, H'H'' = (C' - B')\, (y'' - R \cos.\, \theta' \sin.\, d \sin.\, \mu'' \sin.\, \mu'). \tag{856}$$

If now we put

$$\tan.\, \tfrac{1}{2}\, \psi = \frac{C'}{H'} = \frac{H'}{B'}, \tag{857}$$

we have

$$\frac{C' - B'}{2\, H'} = \tfrac{1}{2} \tan.\, \tfrac{1}{2}\, \psi - \tfrac{1}{2} \cot.\, \tfrac{1}{2}\, \psi$$

$$= \frac{\sin.^2 \tfrac{1}{2}\, \psi - \cos.^2 \tfrac{1}{2}\, \psi}{2 \sin.\, \tfrac{1}{2}\, \psi \cos.\, \tfrac{1}{2}\, \psi} = -\frac{\cos.\, \psi}{\sin.\, \psi} = -\cot.\, \psi = -\frac{y - y'}{x - x'}$$

$$H'' = -\cot.\, \psi\, (y'' - R \cos.\, \theta' \sin.\, d \sin.\, \mu'' \sin.\, \mu'). \tag{858}$$

The change of $H' - A'$ in one second is then $H'' - A''$, so that the number of seconds in which it will decrease by the whole amount of difference $H' - A'$ is

$$t = \frac{H' - A'}{A'' - H''}. \tag{859}$$

176. *Corollary.* It is sufficiently accurate in this example to put

$$\mu'' = 15''$$

$$\sin.\, \mu'' = \quad 5.8617. \tag{860}$$

177. *Corollary.* It is easy to see that ψ is the angle which the

line joining the centre of the shadow with the place makes with the line of reference; it is nearly *the angle from the north point of the sun to the point of contact at the end of the eclipse.*

178. *Problem. To find the limits within which the eclipse is seen in the horizon.*

Solution. In this case the place is nearly in the plane of reference which passes through the earth's centre, and the deviation from this plane may be neglected without much error. If then (fig. 62), S is the earth's centre, AB the path of the centre of the shadow, M the position of the centre at the instant when the eclipse is seen in the horizon at m, the sides of the triangle MSm are

$$\varrho = SM, \; \varrho' = Mm, \; R = Sm.$$

Let $$\eta = m\,S\,m,$$

and we have by (152)

$$\sin.\tfrac{1}{2}\,\eta = \pm\sqrt{\left(\frac{(\varrho' - \varrho + R)\,(\varrho' + \varrho - R)}{4\,\varrho'\,R}\right)}, \qquad (861)$$

in the first computation of which R may be supposed to be the earth's mean radius, or that for the latitude of 45°.

If, then, SC is the line of reference already adopted, we have

$$CSM = \omega'$$
$$CSm = \omega' \pm \eta = \eta', \qquad (862)$$

in which both signs must be used for the two different solutions of the problem.

If now in (fig. 35), L represents the place m of (fig. 62), we have by the right triangle LNP

$$\sin.\,\theta' = \quad \cos.\,\eta'\,\cos.\,d \qquad (863)$$
$$\cot.\,\mu' = -\cot.\,\eta'\,\sin.\,d \qquad (864)$$
$$\lambda' = \mu' - \mu. \qquad (865)$$

179. *Corollary.* The beginning or ending of the eclipse upon the earth, corresponds to the cases of

$$\varrho = \varrho' \pm R, \;\; \eta = O. \qquad (866)$$

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180. *Problem. To find the northern and southern limits of the eclipse upon the earth.*

Solution. For this problem, it is accurate enough to regard the shadow upon the primitive plane of reference as being of uniform width, and the path of its centre as a straight line. If (fig. 62) represents a plane of reference at any height

$$z = R \sin. \varphi, \tag{867}$$

above the original plane, and if AB is one of the bounding lines of the shadow which is drawn parallel to the path of the centre at the distance

$$\varrho' - z \tan. f \tag{868}$$

from this path. If FS is the perpendicular let fall upon AB.

Let
$$P' = FS$$
$$\xi = FSC,$$

FSC being counted negatively, we have

$$\tan. \xi = \frac{y''}{x''} \tag{869}$$

$$P = \text{the perpendicular upon the path}$$

$$= \varrho \cos. (\omega' + \xi) \tag{870}$$

$$P' = P - \varrho' + z \tan. f. \tag{871}$$

Let $\quad CSB = \eta',$

and we have

$$SB = R \cos. \varphi$$

$$\cos. (\xi + \eta') = \frac{P'}{R \cos. \varphi}, \tag{872}$$

we have, then, in (fig. 35)

$$BZP = \eta'$$
$$BL = \varphi,$$

whence the triangle BZP gives by Napier's analogies

$$\tan. \tfrac{1}{2} (B - \mu') = \sin. \tfrac{1}{2} (\varphi - d) \sec. \tfrac{1}{2} (\varphi + d) \cot. \tfrac{1}{2} \eta' \tag{873}$$

$$\tan. \tfrac{1}{2} (B + \mu') = \cos. \tfrac{1}{2} (\varphi - d) \operatorname{cosec.} \tfrac{1}{2} (\varphi + d) \cot. \tfrac{1}{2} \eta' \tag{874}$$

$$\cos. \theta' = \sin. \eta' \cos. \varphi' \operatorname{cosec.} \mu'. \tag{875}$$

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The value of μ may be found from the value of x by inspection of the table, and the value of x is given by the equation

$$x = R \cos. \varphi \sin. \eta' - (\varrho' - z \tan. f) \sin. \xi. \qquad (876)$$

This solution may be corrected by introducing the actual motion of the shadow's centre at the instant, and the motion of the point upon the earth's surface, which is effected by substituting in (869) for y'' and x'', the motion of $y - y'$ and $x - x'$, by which it becomes

$$\tan. \xi = \frac{B''}{A''}. \qquad (877)$$

181. *Corollary.* The phenomena of the central eclipse may be determined by putting

$$\varrho' = 0 \qquad (878)$$

in the various equations.

182. *Problem. To find the instant and amount of greatest obscuration.*

Solution. The instant of greatest obscuration must be when the motion of the axis of the shadow and of the place are neither towards nor from each other, but in parallel lines. In this case the relative motion of the centre of the shadow on the plane of reference is perpendicular to the line drawn from the place, or in other words we then have

$$\xi = -\psi. \qquad (879)$$

Now when ξ has been found for a time near that of greatest conjunction, it changes so slowly, that it is only necessary to find when $-\psi$ has this same value.

But if for any time we have $-\psi$ different from ξ, and denote by P the perpendicular upon the relative path of the centre, we have

$$P = H' \cos. (\psi + \xi), \qquad (880)$$

and the distance by which P the centre must approach the line of reference before it arrives at the point of nearest approach is

$$H' \sin. \psi + H' \cos. (\psi + \xi) \sin. \xi = H' \sin. (\psi + \xi) \cos. \xi. \qquad (881)$$

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Hence the interval of time required for this approach is

$$t = \frac{H' \sin. (\psi + \xi) \cos. \xi}{A''}. \tag{882}$$

The amount of obscuration is proportioned to the distance by which the place is immersed within the penumbra, and is denoted by 12 digits when it is total; that is, when it is immersed by the distance

$$\varrho' \text{ for penumbra} - \varrho' \text{ for shadow} = M. \tag{883}$$

When it is therefore immersed, as in this case, by the quantity

$$\varrho' \text{ for penumbra} - P = N, \tag{884}$$

we have

$$\begin{aligned} n &= \text{the number of digits eclipsed} \\ &= 12 \,.\, \frac{N}{M}. \end{aligned} \tag{885}$$

183. *Corollary.* In the case of an annular eclipse ϱ' in the second term of M must be taken negatively

184. *Corollary.* In the case of the first or last instant of contact, when

$$H' = \varrho' \text{ for penumbra}, \tag{886}$$

we have

$$N = H' [1 - \cos. (\psi - \xi)] = 2 H' \sin.^2 \tfrac{1}{2} (\psi - \xi) \tag{887}$$

and by putting

$$e = \frac{\varrho' \text{ for shadow}}{\varrho' \text{ for penumbra}} \tag{888}$$

we have

$$n = \frac{24}{1 - e} \sin.^2 \tfrac{1}{2} (\psi - \xi). \tag{889}$$

185. *Corollary.* In the case of occultations we have

$$\begin{aligned} f &= 0 \\ \varrho' &= .27227. \end{aligned} \tag{890}$$

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186. *Problem. To compute the longitude of a place from an observed eclipse.*

Solution. By means of an assumed longitude find the approximate Greenwich time of the observation; and compute the eclipse for this time by art. 175. The principal effect of an error in the assumed longitude is to change the Greenwich time, and does not materially affect the value of μ'. If, then, in computing the correction of the time, μ'' is supposed to be zero, the correction obtained becomes one of longitude, to be applied negatively to the eastern longitude.

187. *Problem. To compute the effect of an increase of one second of arc in the moon's relative longitude upon the computed time of an eclipse.*

Solution. By this change in longitude, the moon's shadow is advanced upon the plane of reference by a quantity

$$s = r' \sin. 1'',$$

in a direction which is inclined by an angle $90° - (u - \omega)$ to the line of reference, so that

$$x \text{ and } A' \text{ are increased by } r' \sin. 1'' \cos. (u - \omega),$$

and

$$y, B, \text{ and } C \text{ are increased by } r' \sin. 1'' \sin. (u - \omega).$$

Hence H'^2 will be increased by

$$(C' - B')\, r' \sin. 1'',$$

and H' by

$$\frac{C' - B'}{2\,H'}\, r' \sin. 1'' = - \cot. \psi\, r' \sin. 1'' \sin. (u - \omega), \tag{891}$$

and $H' - A'$ by

$$- [\cot. \psi \sin. (u - \omega) + \cos. (u - \omega)]\, r' \sin. 1''$$

$$= - \frac{\sin. (\psi + u - \omega)}{\sin. \psi}\, r' \sin. 1'', \tag{892}$$

and the corresponding change of time is found from (859).

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188. *Problem. To compute the effect of an increase of one second in the moon's relative latitude upon the computed time of an eclipse.*

Solution. By this increase of latitude the values of x and A are increased by

$$- r' \sin. 1'' \sin. (u - \omega),$$

and y, B and C are increased by

$$r' \sin. 1'' \cos. (u - \omega).$$

Hence H' is increased by

$$- \cot. \psi \, r' \sin. 1'' \cos. (u - \omega),$$

and $H' - A'$ by

$$- \cos. (\psi + u - \omega) \operatorname{cosec.} \psi \, r' \sin. 1'', \tag{893}$$

and the change of time is found by (859).

189. *Problem. To compute the effect of an increase in the moon's semidiameter upon the time of an eclipse.*

Solution. An increase of δs in the moon's semidiameter increases ϱ' for the total shadow and penumbra, and decreases it for the annular phase by about this same amount. Hence H'^2 is increased by

$$(B' + C') \, \delta s$$

and H' is increased by

$$\frac{B' + C'}{2 H'} \delta s = (\tfrac{1}{2} \tan. \tfrac{1}{2} \psi + \tfrac{1}{2} \cot. \tfrac{1}{2} \psi) \, \delta s = \operatorname{cosec.} \psi \, \delta s, \tag{894}$$

and the change of the time is computed by (859).

190. *Problem. To compute the effect of an increase of the moon's parallax upon the time of an eclipse.*

Solution. By an increase of a fractional part $\delta \pi$ in the moon's parallax, the quantities x, y, z are proportionally diminished, the moon's distance from the earth is proportionally diminished, and, to preserve the same apparent semidiameter of the moon, K must be proportionally diminished, and therefore also ϱ'.

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Hence B is diminished by

$$(\varrho' + y)\,\delta\,\pi,$$

C is diminished by

$$(y - \varrho')\,\delta\,\pi,$$

H'^2 is diminished by

$$C'\,(\varrho' + y)\,\delta\,\pi + B'\,(\varrho' - y)\,\delta\,\pi, \tag{895}$$

H' is diminished by

$$\frac{C'+B'}{2\,H'}\,\varrho'\,\delta\,\pi + \frac{C'-B'}{2\,H'}\,y\,\delta\,\pi = \text{cosec.}\,\psi\,\varrho\;\delta\,\pi - \text{cot.}\,\psi\,y\,\delta\,\pi, \tag{896}$$

and A' is diminished by

$$x\,\delta\,\pi.$$

Hence $H' - A'$ is diminished by

$$\begin{aligned}
&\text{cosec.}\,\psi\,\varrho'\,\delta\,\pi - \text{cot.}\,\psi\,y\,\delta\,\pi - x\,\delta\,\pi \\
&= \text{cosec.}\,\psi\,\varrho'\,\delta\,\pi - \varrho\,\text{cot.}\,\psi\,\text{cos.}\,\omega'\,\delta\,\pi - \varrho\,\text{sin.}\,\omega'\,\delta\,\pi \\
&= \text{cosec.}\,\psi\,\delta\,\pi\,[\varrho' - \varrho\,\text{cos.}\,(\psi - \omega')].
\end{aligned} \tag{897}$$

The effect upon the time is computed by (859).

191. Examples.

1. In the solar eclipse of July 28, 1851, to find the position of the line, which is drawn through the earth's centre parallel to the line joining the centres of the sun and moon.

Solution. The following data are taken from the Nautical Almanac and Airy's Lunar Tables with Longstreth's corrections.

Greenw. m. s. t.	☽'s — ☉'s long. $= \lambda$	☽'s — ☉'s lat. $= \beta$	☉'s long. $= l$
0^h	$-1°\ 32'\ 55''.6$	$0°\ 37'\ 16''.0$	$124°\ 45'\ 14''.2$
1	—0 58 24 .0	0 40 39 .2	124 47 37 .7
2	—0 23 51 .0	0 44 2 .2	124 50 1 .2
3	0 10 43 .4	0 47 24 .9	124 52 24 .6
4	0 45 19 .1	0 50 47 .4	124 54 48 .1
5	1 19 56 .1	0 54 9 .7	124 57 11 .6

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Greenw. m. s. t.	☉'s R. A. $= \alpha$	☉'s Dec. $= \delta$	☽'s hor. par. $= \pi$
0^h	127° 6′ 5″.0	19° 5′ 24″.7	60′ 30″.6
1	127 8 32 .6	19 4 50 .2	60 31 .7
2	127 11 0 .0	19 4 15 .7	60 32 .8
3	127 13 27 .3	19 3 41 .2	60 33 .8
4	127 15 54 .6	19 3 6 .6	60 34 .8
5	127 18 21 .8	19 2 32 .1	60 35 .9

$O =$ obliquity of the ecliptic $= 23° 27' 27''.1$

log. $r =$ log. of dist. from sun to earth $= 0.00657$

sidereal time of mean noon $= 8^h\ 22^m\ 13^s.27$

m is thus computed for 0^h from (811)

const.	5.6189
log. r	0.0066
log. sin. π	8.2455
log. m	7.3668

a and d are found from (812 – 817)

λ sin.	8.43181 n	tan.	8.43197 n
β cot.	1.96494		
u tan.	0.39675 n	cosec.	0.03240 n
$\gamma =$	1° 40′ 7″.2	tan.	8.46437
$u =$	— 68° 8′ 40″	γ cos.	9.99982
		m	7.3668
		cosec. 1″	5.3144
$c =$	14″.0	c	1.1454
$c + \gamma =$	1° 40′ 21″.2	cos. l	9.75599 n
		tan. O	9.63742
$u - \omega =$	— 13° 53′ 38″	tan.	9.39333 n
$\omega =$	— 54° 15′ 2″	sin.	9.90933 n
cos. ω	9.76659	sec. δ	0.02457
c	1.1454		1.1454
$\delta - d$	0.9120	$\alpha - a$	1.0793 n
$\delta - d$	8″.2	$\alpha - a$	— 12″.0
δ	19° 5′ 24″.7	α	127° 6′ 5″.0
d	19° 5′ 16″.5	a	127° 6′ 17″.0

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Similar computations for the other dates give

Gr. m. s. t.	a	d	ω	$c+\gamma$
0^h	127° 6′ 17″.0	19° 5′ 16″.5	—54° 15′ 2″	100′ 21″.2
1	127 8 39 .5	19 4 42 .8	—41 15 5	71 19 .2
2	127 11 1 .8	19 4 9 .0	—14 31 4	50 11 .8
3	127 13 24 .0	19 3 35 .1	+26 40 36	48 43 .6
4	127 15 46 .3	19 3 1 .2	+55 41 10	68 13 .5
5	127 18 8 .5	19 2 27 .6	+69 50 6	96 46 .4

2. In the solar eclipse of July 28, 1851, to find the path of the centre of the moon's shadow upon the plane which passes through the earth's centre perpendicular to the line which joins the centres of the sun and moon.

Solution. We have for 0^h by (818 – 821)

	c	1.1454
	sin. ω	9.90933 n
	tan. δ	9.53919
$\omega - \omega' = \quad -3''.9$	$\omega - \omega'$	0.5939 n
$\omega' = -54° \; 14' \; 58''$	cosec. π	1.75447
	sin. $(c+\gamma)$	8.46520
	ϱ	0.21967
	sin. ω'	9.90933 n
	cos. ω'	9.76661
$x = -1.34586$		0.12900 n
$y = \quad .96890$		9.98628

The same computation for the other dates gives

Gr. m. s. t.	x	y
0^h	— 1.34586	0.96890
1	— 0.77694	0.88585
2	— 0.20786	0.80260
3	+ 0.36121	0.71897
4	0.93018	0.63487
5	1.49895	0.55048

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3. In the solar eclipse of July, 28, 1831, to find the umbral and penumbral radii upon the plane of reference of the preceding example.

Solution. Equations (822 – 828) give for 0^h

		m	7.3668
		$\cos.(c+\gamma)$	9.99982
$m\cos.(c+\gamma)$	$=.002326$		7.3666
$1-m\cos.(c+\gamma)$	$=.997674$		9.99899
		r	0.00657
		s'	0.00556
$\log.(\sin. H - K\sin.\Pi)$			7.66669
$\log.(\sin. H + K\sin.\Pi)$			7.66880
for shadow		$\sin.f=\tan.f$	7.66113
for penumbra		$\sin.f=\tan.f$	7.66324
$\text{cosec.}\,\pi$			1.75447
$\cos.(c+\gamma)$			9.99982
z			1.75429
for shadow	$z\tan.f=.26027$		9.41542
for penumbra	$z\tan.f=.26154$		9.41753
K	$=K\sec.f=.27227$		
for shadow	$\varrho'=.01200$		
for penumbra	$\varrho'=.53381$		

Similar computations give for the other dates

Gr. m. s. t.	ϱ' for shadow.	ϱ' for penumbra.
0^h	0.01200	0.53381
1	0.01203	0.53378
2	0.01207	0.53373
3	0.01214	0.53366
4	0.01225	0.53356
5	0.01237	0.53343

F does not perceptibly change its values.

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4. To find the elements for determining the beginning or end of the solar eclipse of July 28, 1851, for any place.

Solution. The value of x is already determined. The values of B, C, E, F, G and H are computed for 0^h from equations (841–846).

	For shadow.	For penumbra.
y	0.96890	0.96890
ϱ'	— 0.01200	0.53381
B	0.95670	1.50271
C	0.98090	0.43509
$d+f$	19° 21′ 1″.8	19° 21′ 6″.4
$d-f$	18° 49′ 31″.2	18° 49′ 26″.6
sec. f	0.00000	0.00000
log. E	9.97612	9.97613
log. F	9.97475	9.97475
log. G	9.50878	9.50874
log. H	9.52028	9.52031

Right ascension of Greenwich meridian $0^h = 8^h\ 22^m\ 13^s.27$

$= 125°\ 33'\ 19''.0$

$a = 127\ \ 6\ \ 17\ .0$

$\mu = -\ \ 1\ \ 32\ \ 58\ .0$

These values may be computed in the same way for other dates, and being interpolated by differences for every five minutes, may be arranged in a table as follows.

For penumbra.

Greenw. m. s. t.	x	B	C	log. E 9.97	log. F 9.97	log. G 9.50	log. H 9 5	μ
$0^h\ 0^m$	—1.34586	1.50271	.43509	613	475	874	2031	— 1° 32′ 58″.0
5	—1.29845	1.49580	.42818	13	75	872	2029	0 17 57 .6
10	—1.25104	1.48889	.42127	13	75	871	2027	+ 0 57 2 .8
15	—1.20363	1.48197	.41436	13	76	869	2026	2 12 3 .3
20	—1.15622	1.47505	.40744	14	76	867	2024	3 27 3 .7
25	—1.10881	1.46813	.40052	14	76	866	2022	4 42 4 .1
30	—1.06140	1.46121	.39360	14	76	864	2021	5 57 4 .6

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Greenw. m. s. t.	x	B	C	log. E 9.97	log. F 9.97	log. G 9.50	log. H 9.5	μ
$0^h 35^m$	—1.01399	1.45429	.38668	614	476	863	2019	7° 12′ 5″.0
40	—0.96658	1.44737	.37976	15	76	861	2017	8 27 5 .5
45	—0.91917	1.44044	.37284	15	77	859	2016	9 42 5 .9
50	—0.87176	1.43351	.36592	15	77	857	2014	10 57 6 .4
55	—0.82435	1.42658	.35900	16	77	855	2012	12 12 6 .8
1 0	—0.77694	1.41965	.35207	16	77	853	2011	13 27 7 .3
5	—0.72952	1.41272	.34514	16	77	851	2009	14 42 7 .8
10	—0.68210	1.40578	.33821	16	77	850	2007	15 57 8 .2
15	—0.63468	1.39884	.33128	17	78	848	2006	17 12 8 .7
20	—0.58726	1.39190	.32435	17	78	846	2004	18 27 9 .2
25	—0.53983	1.38496	.31742	17	78	845	2002	19 42 9 .6
30	—0.49241	1.37802	.31049	17	78	843	2001	20 57 10 .1
35	—0.44499	1.37108	.30356	17	78	842	1999	22 12 10 .5
40	—0.39756	1.36413	.29663	17	78	840	1997	23 27 11 .0
45	—0.35014	1.35718	.28969	18	79	838	1996	24 42 11 .5
50	—0.30272	1.35023	.28275	18	79	837	1994	25 57 12 .0
55	—0.25529	1.34328	.27581	18	79	835	1992	27 12 12 .4
2 0	—0.20786	1.33633	.26887	18	79	833	1990	28 27 12 .9
5	—0.16043	1.32937	.26192	18	79	831	1988	29 42 13 .4
10	—0.11300	1.32241	.25497	18	79	830	1987	30 57 13 .8
15	—0.06558	1.31545	.24802	19	80	828	1985	32 12 14 .3
20	—0.01816	1.30848	.24106	19	80	826	1983	33 27 14 .8
25	+0.02927	1.30151	.23410	19	80	825	1982	34 42 15 .2
30	0.07669	1.29454	.22714	19	80	823	1980	35 57 15 .7
35	0.12411	1.28757	.22018	19	81	821	1978	37 12 16 .2
40	0.17153	1.28059	.21323	19	81	820	1977	38 27 16 .6
45	0.21895	1.27361	.20624	20	81	818	1975	39 42 17 .1
50	0.26637	1.26663	.19927	20	81	816	1973	40 57 17 .6
55	0.31379	1.25965	.19229	20	82	814	1972	42 12 18 .0
3 0	0.36121	1.25267	.18531	20	82	812	1970	43 27 18 .5
5	0.40863	1.24568	.17832	20	82	810	1968	44 42 19 .0
10	0.45605	1.23868	.17133	20	82	809	1967	45 57 19 .4
15	0.50347	1.23169	.16434	21	83	807	1965	47 12 19 .9
20	0.55088	1.22467	.15734	621	483	805	1963	48 27 20 .3

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Greenw. m. s. t.	x	B	C	log. E 9.97	log. F 9.97	log. G 9.50	log. H 9.5	μ
$3^h 25^m$	0.59830	1.21764	.15034	621	483	804	1962	49° 42′ 20″.8
30	0.64571	1.21061	.14334	21	83	802	1960	50 57 21 .3
35	0.69312	1.20358	.13634	21	84	800	1958	52 12 21 .7
40	0.74054	1.19655	.12934	21	84	799	1957	53 27 22 .2
45	0.78795	1.18952	.12234	22	84	797	1955	54 42 22 .6
50	0.83536	1.18249	.11533	22	84	795	1953	55 57 23 .1
55	0.88277	1.17546	.10832	22	85	793	1952	57 12 23 .6
4 0	0.93018	1.16843	.10131	22	85	791	1950	58 27 24 .0
5	0.97758	1.16140	.09430	22	85	789	1948	59 42 24 .5
10	1.02498	1.15436	.08729	23	85	787	1947	60 57 24 .9
15	1.07238	1.14732	.08027	23	85	786	1945	62 12 25 .4
20	1.11978	1.14028	.07325	23	86	784	1943	63 27 25 .8
25	1.16718	1.13324	.06623	23	86	782	1942	64 42 26 .3
30	1.21458	1.12620	.05921	23	86	780	1940	65 57 26 .7
35	1.26198	1.11916	.05219	24	86	779	1938	67 12 27 .2
40	1.30937	1.11211	.04517	24	86	777	1937	68 27 27 .6
45	1.35637	1.10506	.03814	24	87	775	1935	69 42 28 .1
50	1.40416	1.09801	.03111	24	87	774	1933	70 57 28 .5
55	1.45156	1.09096	.02408	25	87	772	1932	72 12 29 .0
5 0	1.49895	1.08391	.01705	625	487	770	1930	73 27 29 .4

$$x'' = \quad 0.0001580$$

$$y'' = -0.0000232$$

Gr. m. s. t.	For shadow. B	C
$1^h 25^m$	.83914	.86322
30	.83220	.85629
35	.82526	.84936
40	.81832	.84243
45	.81138	.83549
50	.80443	.82855
55	.79748	.82151
2 0	.79053	.81467
5	.78358	.80772

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Gr. m. s. t.	For shadow. B	C
$2^h\ 10^m$	.77662	.80077
15	.76966	.79382
20	.76269	.78686
25	.75572	.77990
30	.74874	.77294
35	.74176	.76598
40	.73477	.75901
45	.72778	.75204
50	.72079	.74507
55	.71379	.73809
3 0	.70679	.73111
5	.69979	.72413
10	.69279	.71715
15	.68579	.71016
20	.67879	.70317
25	.67178	.69618
30	.66477	.68919
35	.63776	.68219
40	.65075	.67519
45	.64374	.66819

x and μ are given in the penumbral table, and log. E and log. F are sensibly the same for shadow and penumbra, while log. G and log. H for shadow are obtained from the corresponding values for penumbra by increasing log. G by 0.00003, and decreasing log. H by this same quantity.

5. To compute the phases of the eclipse of July 28, 1851, for Dantzic.

Solution. For Dantzic

$$\text{the latitude} = 54°\ 20'\ 18''$$
$$\text{the longitude} = 1^h\ 14^m\ 41^s.5$$
$$= 18°\ 40'\ 22''.5$$

Red. of latitude	10′ 53″
θ'	54° 9′ 25″

then by (750)

		sin.² lat.	9.8196
		$\frac{1}{300}$	7.5229
δR	$=$ 0.002201		7.3425
R	$=$ .997799		9.99904
sin. θ'			9.90882
cos. θ'			9.76758
R sin. $\theta' = h$			9.90786
R cos. $\theta' = h'$			9.76642

I. For the beginning, computing now for $2^h\ 15^m$ by the equations (830–859) we have

μ	32° 12′ 14″.3
γ'	18° 40′ 22″.5
μ'	50° 52′ 36″.8

h	9.90786		9.90786
E	9.97619	F	9.97480
$h\,E$	9.88405	$h\,F$	9.88266
h'	9.76662		9.76662
G	9.50828	H	9.51985
cos. μ'	9.80002		9.80002
$h'\,G$ cos. μ'	9.07492	$h\,H$ cos. μ'	9.08649
B	1.31545	$-C$	$-$ .24802
$-h\,E$	$-$ .76568	$h\,F$	.76323
$h'\,G$ cos. μ'	.11883	$-h'\,H$ cos. μ'	$-$.12204
B'	.66860	C'	.39317
		C'	9.59458
		B'	9.82517
		$B'C'$	9.41975
H' $=$	$-$.51271		9.70987n

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		h'	9.76662
$H' =$	$-.51271$	sin. μ'	9.88974
$x' =$	$.45327$		9.65636
$x =$	$-.06558$		
$A' =$	$-.51885$	h' cos. μ'	9.5666
$H' - A' =$	$.00614$	const.	5.8617
h' cos. μ' sin. $15''$ $=$	$.0000268$		5.4283
x''	$.0001580$	x'	9.6564
A''	$.0001312$	const.	5.8617
y''	$-.0000232$	sin. d	9.5142
x' sin. d sin. $15''$	$.0000108$		5.0323
$y'' - x'$ sin. d sin. $15''$	$-.0000340$		5.5315n
$\frac{1}{2}\psi = -37° 28' 58''$		tan. $\frac{1}{2}\psi$	9.88471n
$\psi = -74° 57' 56''$		$-$cot. ψ	9.4291
$H'' =$	$.0000091$		4.9606n
$A'' - H''$	$.0001403$		6.1471
$H = A'$			7.7882
$t =$	$43''.8$		1.6411

Gr. time of beg. $=$	$2^h\ 15'\ 43''.8$
long.	1 14 41 .5
Dant. time of beg. $=$	3 30 25 .3

II. For the end, computing at $4^h\ 17^m$

μ	$62°\ 42'\ 25''.6$
λ'	18 40 22 .5
μ'	81 22 48 .1

h	9.90786		9.90786
E	9.97623	F	9.97485
$h\,E$	9.88409	$h\,F$	9.88271
h'	9.76662		9.76662
G	9.50785	H	9.51944
cos. μ'	9.17575		9.17575
$h'\,G$ cos. μ'	8.45022	$h'\,H$ cos. μ'	8.46181

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B	1.14450	$-C$	$-.07746$
$-hE$	$-.76575$	hF	.76332
$h' G \cos. \mu'$	.02820	$-h' H \cos. \mu'$	$-.02896$
B'	.40695	C'	.65690
		C'	9.81750
		B'	9.60955
		$B'C'$	9.42705
$H' =$	.51704		9.71352
		h'	9.76662
		$\sin. \mu'$	9.99507
$x' =$	.57769		9.76169
$x =$	1.09134		
$A' =$	.51365	$h' \cos. \mu'$	8.9424
$H' - A' =$	.00339	const.	5.8617
$h' \cos. \mu' \sin. 15''$	.0000064		4.8041
x''	.0001580	x'	9.7617
A''	.0001516	$\sin. 15'' \sin. d$	5.3759
$x'' \sin. d \sin. 15''$	.0000137		5.1376
y''	$-.0000232$		
$y'' - x' \sin. d \sin. 15''$	$-.0000369$		5.5670 n
$\frac{1}{2}\psi = 51° 47' 40''$		$\tan. \frac{1}{2}\psi$	0.10398
$\psi = 103\ \ 35\ \ 20$		$-\cot. \psi$	9.3833
$H'' =$	$-.0000089$		4.9503 n
$A'' - H'' =$	.0001605		6.2054
$H' - A'$			7.5302
$t =$	$21''.1$		1.3248

Greenw. time of end. $4^h\ 17^m\ 21^s.1$

long. 1 14 41 .5

Dantzic time of end. 5 32 2 .6

III. For beginning of total phase, computing at $3^h\ 17^m$.

		μ	47° 42′ 20″.1
		λ'	18 40 22 .5
		μ'	66 22 42 .6
h	9.90786		9.90786
E	9.97621	F	9.97483
$h\,E$	9.88407	$h\,F$	9.88269
h'	9.76662		9.76662
G	9.50809	H	9.51961
$\cos.\ \mu'$	9.60281		9.60281
$G\,h'\cos.\ \mu'$	8.87752	$H\,h'\cos.\ \mu'$	8.88904
B	.68299	$-C$	$-$ .70736
$-h\,E$	$-$ .76572	$h\,F$	.76328
$G\,h'\cos.\ \mu'$	.07542	$-h'\,H\cos.\ \mu'$	$-$ .07745
B'	$-$ .00731	C'	$-$ .02153
		C'	8.33304 n
		B'	7.86392 n
		$B'\,C'$	6.19696
$H' =$	$-$ .01255		8.09848 n
		h'	9.76662
$x =$	.52243	$\sin.\ \mu'$	9.96199
$x' =$	.53531		9.72861
$A' =$	$-$ .01288	$h'\cos.\ \mu'$	9.3694
$H' - A' =$	.00033	const.	5.8617
$h'\cos.\ \mu'\sin.\ 15''$	.0000170		5.2311
x''	.0001580	const. $\sin.\ d$	5.3759
A''	.0001410	x'	9.7286
$x'\sin.\ d\sin.\ 15''$	.0000127		5.1045
y''	$-$ .0000232		

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$y'' - x' \sin. d \sin. 15''$	$-$ .0000359		5.5551 n
$\frac{1}{2}\psi$	59° 46′ 14″	tan.	0.23456
ψ	119° 32′ 28″	$-$ cot. ψ	9.7534
H''	$-$ .0000203		5.3085 n
$A'' - H''$	.0001613		6.2076
		$H' - A'$	6.5185
t		$2^s.0$	0.3109

Greenwich time of beg.	3^h 17^m $2^s.0$
long.	1 14 41 .5
Dantzic time of beg.	4 31 43 .5

IV. For end of total phase, computing at 3^h 20^m.

μ	48° 27′ 20″.3
λ'	18 40 22 .5
μ'	67 7 42 .8

h	9.90786		9.90786
E	9.97621	F	9.97483
$h\,E$	9.88407	$h\,F$	9.88269
h'	9.76662		9.76662
G	9.50808	H	9.51960
cos. μ'	9.58958		9.58958
$G\,h'$ cos. μ'	8.86428	$H\,h'$ cos. μ'	8.87580
B	.67879	$-\,C$	$-$ .70317
$-\,h\,E$	$-$ .76572	$h\,F$	.76328
$G\,h'$ cos. μ'	.07316	$-\,H\,h'$ cos. μ'	$-$ .07513
B'	$-$ .01377	C'	$-$ .01502
		C'	8.17667 n
		B'	8.13893 n
		$B'\,C'$	6.31560
$H' =$	.01438		8.15780

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		h'	9.76662
x	.55088	sin. μ'	9.96443
$x' =$	.53833		9.73105
A'	.01255	h' cos. μ'	9.3562
$H' - A'$	.00183	const.	5.8617
h' cos. μ' sin. 15″	.0000338		5.2179
x''	.0001580	const. sin. d	5.3759
A''	.0001242	x'	9.7311
x' sin. d sin. 15″	.0000128		5.1070
y''	—.0000232		
y''—x'sin. $d$sin.15″	—.0000360		5.5563$_n$
$\frac{1}{2}\psi$	133° 45′ 21″	tan.	0.01887$_n$
ψ	267 30 42	—cot.	8.6381$_n$
H''	.00000016		4.1944
$A'' - H''$	.0001226		6.0885
		$H' - A'$	7.2625
	t	14″.9	1.1760

Greenwich time of ending	3^h 20′ 14″.9
long.	1 14 41 .5
Dantzic time of ending	4 34 56 .4.

6. To compute the places for which the solar eclipse of July 28, 1851, is visible in the horizon at 2^h 0^m, Greenwich mean solar time.

Solution. By formulas (861–866) we find for lat. (45°)

$R =$	.99833	(ar. co.)	0.00073
$\varrho' =$	.53373	log. $\frac{1}{4}$	9.39794
$\varrho =$	.82910	(ar. co.)	0.08139
$\varrho' - \varrho + R =$	.70296		9.84693
$\varrho' + \varrho - R =$	.36450		9.56170
$\sin.^2 \frac{1}{2}\eta$			8.88869

$\frac{1}{2}y$	16° 9′ 9	sin.	9.44434
η	32 18 18		
ω'	—14 31 4		

1st	$\eta' =$	$17° 47' 14''$	cos. 9.97873	cot. 0.49374
	d		cos. 9.97548	sin. 9.51411
	$\theta' =$	64 9	sin. 9.95421	
	$\mu' =$	135 31		cot. 0.00785n
	$\mu =$	28 27		
	$\lambda' =$	107 4		
	$\theta =$	64 18		
2d	$\eta' =$	—46 49 22	cos. 9.83522	cot. 9.97235n
	d		cos. 9.97548	sin. 9.51411
	$\theta' =$	40 18	sin. 9.81070	
	$\mu' =$	252 58		9.48646
	$\mu =$	28 27		
	$\lambda' =$	224 31		
	$\theta =$	40 29.		

7. *Problem. To find a place upon the southern limits of the eclipse of July* 28, 1851, *at an angular height of* 40° *from the plane of reference.*

Solution. By equations (867–877) we find

y''			5.3655n
x''			6.1987
ξ	— 8° 21′	tan.	9.1668n
at 2^h ω'	—14 31	ϱ	9.9186
$\omega' + \xi$	—22 52	cos.	9.9644
P	.7638		9.8830
ϱ'	.5337	R for 45°	9.9993
		sin. 40°	9.8081
		tan. f	7.6632

z tan. f	.0030				7.4706
P'	.2331				9.3676
d	19° 4′			R (ar. co.)	0.0007
φ	40			sec.	0.1157
$\xi + \eta'$	72 15			cos.	9.4840
η'	80 36				
$\frac{1}{2}\eta'$	40 18	cot.	0.0716		0.0716
$\frac{1}{2}(\varphi + d)$	29 32	sec.	0.0604	cosec.	0.3072
$\frac{1}{2}(\varphi - d)$	10 28	sin.	9.2593	cos.	9.9927
$\frac{1}{2}(B - \mu')$	13 50	tan.	9.3913		
$\frac{1}{2}(B + \mu')$	66 58			tan.	0.3715
μ'	53 8			cosec.	0.0969
φ				cos.	9.8843
η'				sin.	9.9941
θ'	19° 6′			cos.	9.9753
R cos. φ sin. η'	.7546				9.8777
				$\varrho' - z$ tan. f	9.7249
				sin. ξ	9.1620
$(\varrho' - z$ tan. $f)$ sin. ξ	.0771				8.8869
x	.6775				
whence μ	52° 36′				
λ'	0 32				

Whence a second approximation may be made with greater accuracy for $3^h\ 30^m$.

		R	9.9999	R	9.9999
		cos. θ'	9.9753	cos. θ'	9.9753
		cos. μ'	9.7781	sin. μ'	9.7031
		sin. 15″	5.8617	sin. 15″	5.8617
h cos. μ' sin. 15″	.0000412		5.6150	sin. d	9.5141
x''	.0001580	x sin. 15″ sin. d	.00000113		5.0541
A''	.0001168	3″	—.00000232		
		B''	—.00000345		5.5378 n
		A''			6.0674

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ξ	$-16° 27'$			tan.	9.4700n
ω'	43 39			ϱ	9.9710
$\omega' + s$	27 12			cos.	9.9491
P	.8320				9.9201
ϱ'	.5336				
z tan. f	.0030				
P'	.3014				9.4791
				$(R \cos. \varphi)^{-1}$	0.1158
$\xi + \eta'$	$66° 50'$			cos.	9.5949
η'	83 17				
$\frac{1}{2} \eta'$	41 38	cot.	0.0512		0.0512
$\frac{1}{2} (\varphi + d)$		sec.	0.0604	cosec.	0.3072
$\frac{1}{2} (\varphi - d)$		sin.	9.2593	cos.	9.9927
$\frac{1}{2} (B - \mu')$	$13° 13'$	tan.	9.3709		
$\frac{1}{2} (B + \mu')$	65 59			tan.	0.3511
μ'	52 46			cosec.	0.0990
φ				cos.	9.8843
η'				sin.	9.9970
θ'	$17° 8'$			cos.	9.9803
θ	17 14				
$R \cos. \varphi \sin. \eta'$	.7607		9.8812		
		$\varrho' - z$ tan. f	9.7248		
		sin. ξ	9.4521		
$(\varrho' - z \tan. f) \sin. \xi$	.1503		9.1769		
x	.6104				
μ	$50° 1'$				
λ'	2 45				

8. To find the instant and amount of greatest obscuration in the total eclipse of July 28, 1851, for Dantzic.

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Solution. From the computation for $3^h\ 17^m$ by (879–889) we have

		B''	5.5551 n	
		A''	6.1045	
ξ	$-15^\circ\ 46'$		9.4506 n	
ψ	119 32			
$\psi+\xi$	103 46	sin.	9.9873	cos. 9.3765
H'			8.0985	8.0985
cos. ξ			9.9833	
A'' (ar. co.)			3.8955	
t		$92^s.2$	1.9646	
Gr. t. of gr. obs.	$3^h\ 18^m\ 32\ .2$			
long.	1 14 41 .5			
Dantzic t.	4 33 13 .7			
		ϱ' for penumbra	.5337	
		ϱ' for shadow	.0121	
		M	.5216	9.7173
		P	.0030	7.4750
		N	.5307	9.7248
	12			1.0792
digits eclipsed $=$	12.2			1.0867

9. To compute the longitude of the Cambridge Observatory from the solar eclipse of July 28, 1851, the beginning of which was observed at $19^h\ 49^m\ 35^s.3$.

Solution. The longitude of Cambridge being about $4^h\ 44^m\ 30^s$, the Greenwich time is not far from $0^h\ 34^m$, for which time the following computation is made.

The latitude	$=$	$42^\circ\ 22'\ 49''$	
assumed longitude	$=$	$-71\ \ 7\ 30$	
reduction of lat.		11 25	
θ'		42 11 24	
sin. lat.			9.6574
$\frac{1}{300}$			7.5229
$\delta R =$		.001515	7.1803

R	.998485		9.99933
sin. θ'			9.82711
cos. θ'			9.86977
h			9.82644
h'			9.86910
		μ	6° 57′ 5″.0
		λ'	—71 7 30
		μ'	—64 10 25
h	9.82644		9.82644
E	9.97614	F	9.97476
$h E$	9.80258	$h F$	9.80120
h'	9.86910		9.86910
G	9.50863	H	9.52019
cos. μ'	9.63913		9.63913
$h' G$ cos. μ'	9.01686	$h' H$ cos. μ'	9.02842
B	1.45567	$-C$	— .38806
$-h E$	— .63471	$h F'$	.63270
$h' G$ cos. μ'	.10396	$-h' H$ cos. μ'	— .10676
B'	.92492	C'	.13788
		C'	9.13950
		B'	9.96610
		$B' C'$	9.10560
H'	— .35711		9.55280 n
		h'	9.86910
x	—1.02347	sin. μ'	9.95430 n
x'	— .66589		9.82340 n
A'	— .35758		
$H' - A'$	.00047		6.6721
$\frac{1}{2}\psi$	—21° 6′ 42″	tan.	9.58670 n
ψ	—42 13 24	—cot.	.0422
x''	.0001580	y''	5.3655 n
$-y''$ cot. ψ	— .0000256		5.4077 n
$x'' - y''$ cot. ψ	.0001324		6.1219

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corr. of long.	$-3^s.5$	0.5502
assumed time	$-4^h\ 40'\ 30$	
computed	$-4\ \ 40\ \ 33.6$	

10. To compute the effect of changes in the moon's relative longitude and latitude, semidiameter and horizontal parallax, upon the time of the beginning of the eclipse of July 28, 1851, for Dantzic.

Solution. By equations (892–897) we find

$\psi = -74^\circ 57'.9$	cosec. $0.0151\,n$		
$A'' - H''$	6.1471		
$-7378^s. = \dfrac{\text{cosec.}\,\psi}{A''-H''}$	$3.8680\,n$		
r'	1.7544		
sin. $1''$	4.6856		
$\dfrac{\text{cosec.}\,\psi}{A''-H''}\, r'$ sin. $1''$	$0.3080\,n$		$0.3080\,n$
$u - \omega = \quad -13^\circ 55'.6$			
$\psi + u - \omega = -88\ \ 53\ .3$ sin.	$9.9999\,n$		cos. 8.2866
$-\dfrac{(892)}{A''-H''} = 2^s.032$	0.3079	$-\dfrac{(893)}{A''-H''} = -0^s.039$	$8.5946\,n$
$\omega' \quad = - \quad 4^\circ 48'.7 \quad \varrho$	9.8946		
$\psi - \omega' = -70 \quad 9\ .2$ cos.	9.5308		
$\varrho \cos.\,(\psi - \omega') \ = .2663$	9.4254		
$\varrho' \qquad = .5337$		$\dfrac{\text{cosec.}\,\psi}{A''-H''}$	$3.8680\,n$
$\varrho' - \varrho \cos.(\psi . \omega') = .2674$			9.4272
$\dfrac{(897)}{(A''-H'')\,\delta\pi} = -\ 1973$			$3.2952\,n$

Hence the changes of the time of beginning for a change of $\delta\lambda$ and $\delta\beta$ expressed in seconds of arc in the moon's relative longitude and

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latitude, a change δs in the moon's semidiameter, and a change of a fractional part $\delta \pi$ in the moon's horizontal parallax are respectively

$$-2^s.032\ \delta\lambda$$
$$0^s.039\ \delta\beta$$
$$-7378^s\ \delta s$$
$$-1973^s\ \delta\pi.$$

11. Compute the end of the solar eclipse of July 28, 1852, for Washington, Paris, Göttingen, Rome and Königsberg.

The latitudes and longitudes of these places are as follows.

	Latitude.	Longitude.
Washington	38° 53′ 34″	$18^h\ 51^m\ 48^s$
Paris	48 50 13	0 9 21 .5
Göttingen	51 31 48	0 39 46 .5
Rome	41 53 52	0 49 54 .7
Königsberg	54 42 50	1 22 0 .5

Ans. The times of beginning and end of the general eclipse are as follows.

	Beginning.	End.
Washington	$19^h\ 21^m\ 16^s.5$	$20^h\ 50^m\ 24^s.7$
Paris	2 21 0 .4	4 30 52 .4
Göttingen	2 53 42 .3	5 0 14 .4
Rome	3 24 27 .3	3 24 32 .7
Königsberg	3 38 20 .1	5 38 48 .3

For the total phase we have

	Beginning.	End.
Königsberg	$4^h\ 39^m\ 10^s.9$	$4^h\ 42^m\ 0^s.8$.

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TABLE I.

Log. n	Corr.	Log. n	Corr.	Log. n	Corr.	Log. n	Corr.	Log. m	Corr.
0.00	0	2.360	49	2.660	97	2.960	192	3.260	384
0.10	0	2.370	50	2.670	99	2.970	197	3.270	394
0.20	0	2.380	51	2.680	101	2.980	202	3.280	403
0.30	0	2.390	52	2.690	103	2.990	207	3.290	413
0.40	0	2.400	53	2.700	106	3.000	211	3.300	423
0.50	1	2.410	54	2.710	108	3.010	216	3.310	433
0.60	1	2.420	55	2.720	111	3.020	221	3.320	443
0.70	1	2.430	56	2.730	113	3.030	227	3.330	453
0.80	1	2.440	58	2.740	116	3.040	232	3.340	464
0.90	2	2.450	59	2.750	119	3.050	238	3.350	475
1.00	2	2.460	61	2.760	122	3.060	243	3.360	486
1.10	2	2.470	62	2.770	124	3.070	249	3.370	497
1.20	3	2.480	64	2.780	127	3.080	255	3.380	508
1.30	4	2.490	65	2.790	130	3.090	261	3.390	520
1.40	6	2.500	67	2.800	133	3.100	267	3.400	532
1.50	7	2.510	69	2.810	136	3.110	273	3.410	545
1.60	9	2.520	70	2.820	139	3.120	279	3.420	558
1.70	11	2.530	72	2.830	143	3.130	286	3.430	571
1.80	14	2.540	73	2.840	146	3.140	292	3.440	584
1.90	17	2.550	75	2.850	150	3.150	299	3.450	598
2.00	21	2.560	77	2.860	153	3.160	306	3.460	612
2.10	26	2.570	78	2.870	157	3.170	312	3.470	626
2.20	33	2.580	80	2.880	160	3.180	320	3.480	641
2.30	42	2.590	82	2.890	164	3.190	328	3.490	657
2.300	42	2.600	84	2.900	167	3.200	335	3.500	672
2.310	43	2.610	86	2.910	171	3.210	343	3.510	688
2.320	44	2.620	88	2.920	175	3.220	351	3.520	704
2.330	45	2.630	90	2.930	179	3.230	360	3.530	720
2.340	47	2.640	92	2.940	184	3.240	368	3.540	737
2.350	48	2.650	94	2.950	189	3.250	376	3.550	754

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TABLE II.

	$\frac{1}{100} R$												
D	0	2	4	6	8	10	12	14	16	18	20	22	24
0°	0	.0	.0	0	.0	.0	.0	.0	.0	.0	.0	.0	.0
1	0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.2	.2	.2
2	0	.0	.0	.0	.1	.1	.1	.2	.2	.3	.3	.4	.5
3	0	.0	.0	.0	.1	.1	·2	.2	.3	.4	.5	.6	.7
4	0	.0	.0	.1	.1	.2	.2	.3	.4	.5	.7	.8	1.0
5	0	.0	.0	.1	.1	.2	.3	.4	.5	.7	.8	1.0	1.2
6	0	.0	.0	.1	.2	.3	.4	.5	.6	.8	1.0	1.2	1.5
7	0	.0	.0	.1	.2	.3	.4	.6	.8	1.0	1.2	1.4	1.7
8	0	.0	.1	.1	.2	.3	.5	.7	.9	1.1	1.3	1.6	1.9
9	0	.0	.1	.1	.2	.4	.5	7	.9	1.2	1.5	1.8	2.2
10	0	.0	.1	.1	3	.4	.6	.8	1.0	1.3	1.7	2.0	2.4
11	0	.0	.1	.2	3	.5	.7	.9	1.2	1.5	1.8	2.2	2.6
12	0	.0	.1	.2	.3	.5	.7	1.0	1.3	1.6	2.0	2.4	2.8
13	0	.0	.1	.2	.3	.5	.8	1.1	1.4	1.7	2.1	2.5	3.0
14	0	.0	.1	.2	.4	.6	.8	1.1	1.4	1.8	2.3	2.8	3.3
15	0	.0	.1	.2	.4	.6	.9	1.2	1.5	1.9	2.4	2.9	3.5
16	0	.0	.1	.2	.4	.6	.9	1.3	1.7	2.1	2.6	3.1	3.7
17	0	.0	.1	.2	.4	.7	1.0	1.4	1.7	2.2	2.7	3.3	3.9
18	0	.0	.1	.3	.5	.7	1.0	1.4	1.8	2.3	2.8	3.4	4.1
19	0	.0	.1	.3	.5	.8	1.1	1.5	1.9	2.4	3.0	3.6	4.3
20	0	.0	1	.3	.5	.8	1.1	1.5	2.0	2.5	3.1	3.8	4.5
21	0	.0	.1	.3	5	.8	1.2	1.6	2.1	2.6	3.2	3 9	4.7
22	0	.0	.1	.3	.5	.8	1.2	1.7	2.2	2.7	3.4	4.1	4.8
23	0	.0	.1	.3	.6	.9	1.3	1.7	2.2	2.8	3.5	4.2	5.0
24	0	.0	.1	.3	.6	.9	1.3	1.8	2.3	2.9	3.6	4.4	5.2
25	0	0	.1	.3	.6	.9	1.3	1.8	2.4	3.0	3 7	4.5	5.4
26	0	.0	.1	.3	.6	1.0	1.4	1.9	2.5	3.1	3 8	4.6	5.5

TABLE III.

δh or δD	
Log.	Sec.
0.0	0.00000
1.0	0.00000
2.0	0 00000
3.0	0.00000
3.1	0.00001
3.2	0.00001
3.3	0.00002
3.4	0.00003
3.5	0.00005
3.6	0.00008
3.7	0.00013
3.8	0.00020

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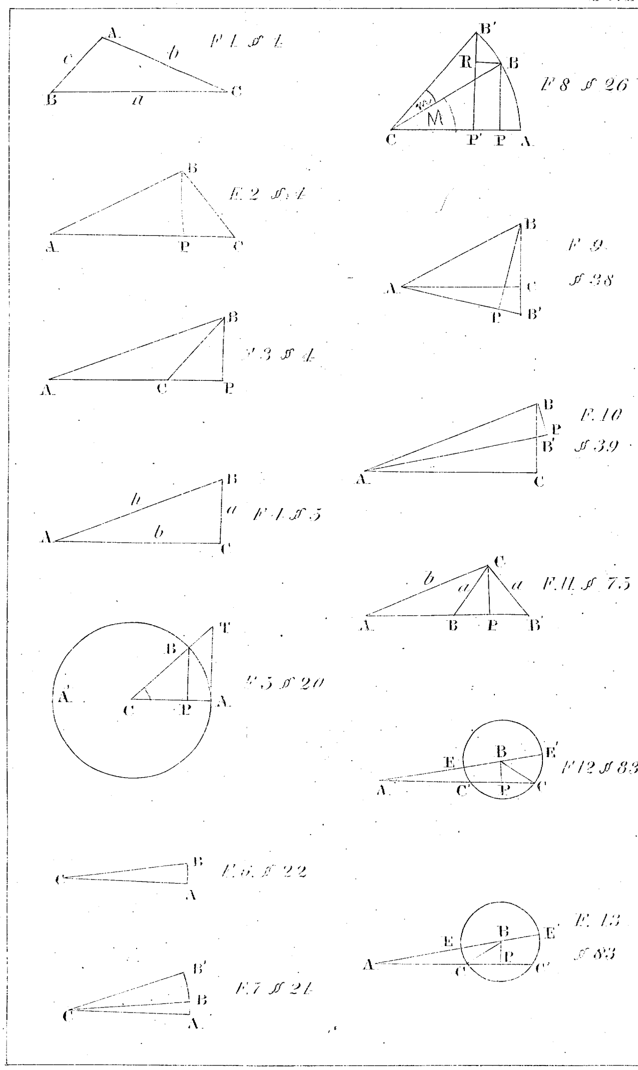

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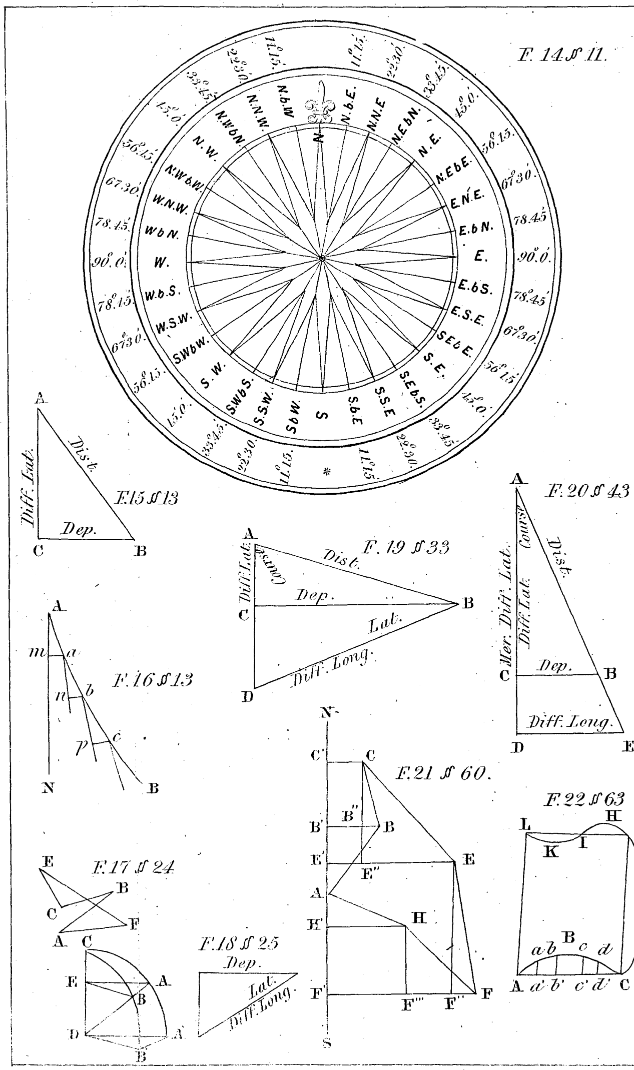

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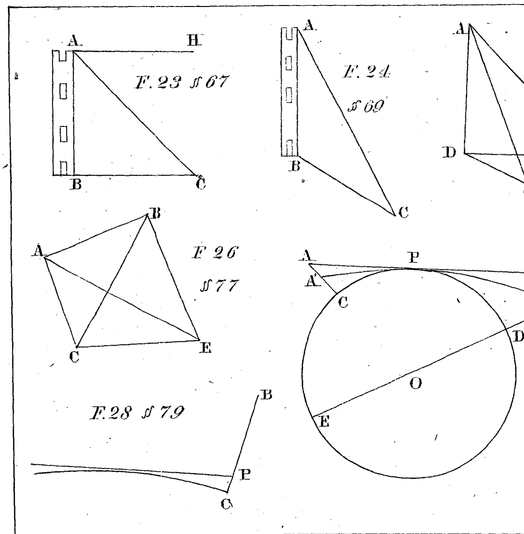

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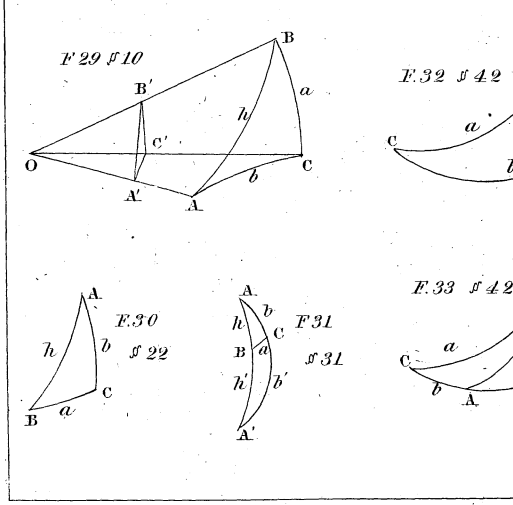

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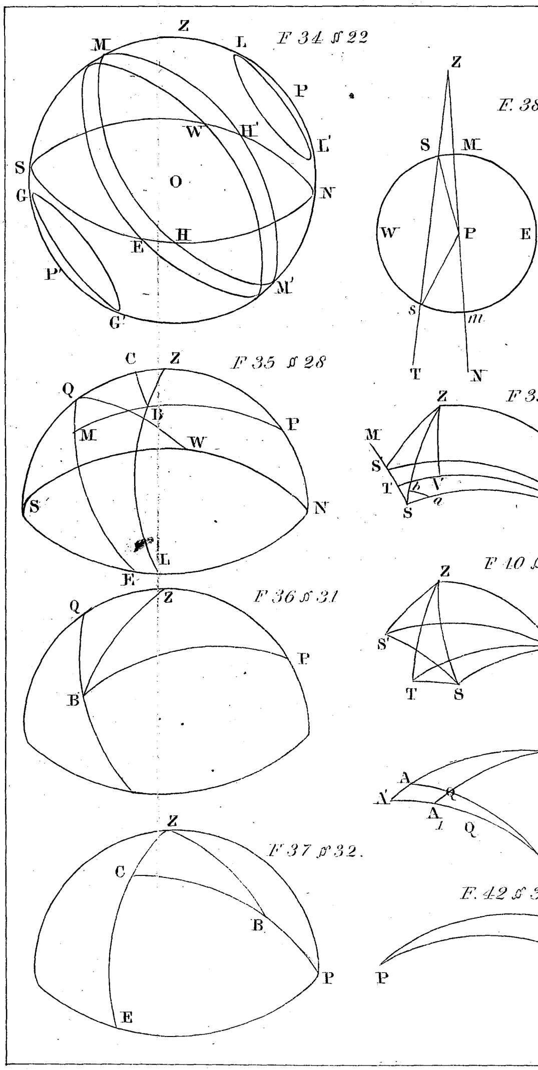

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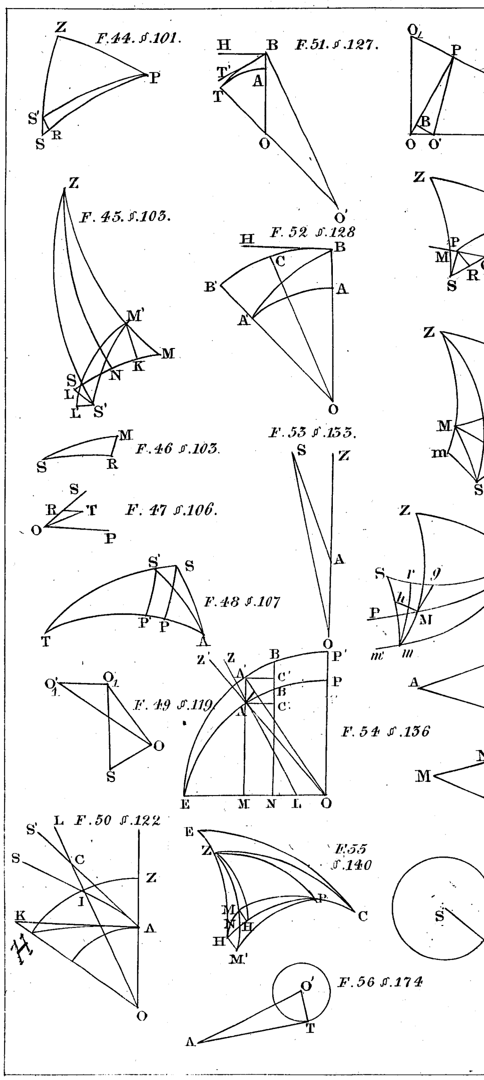

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